

On the total length of the random minimal directed spanning tree

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Abstract

In Bhatt and Roy's minimal directed spanning tree (MDST) construction for a random partially ordered set of points in the unit square, all edges must respect the "coordinatewise" partial order and there must be a directed path from each vertex to a minimal element. We study the asymptotic behaviour of the total length of this graph with power weighted edges. The limiting distribution is given by the sum of a normal component away from the boundary and a contribution introduced by the boundary effects, which can be characterized by a fixed point equation, and is reminiscent of limits arising in the probabilistic analysis of certain algorithms. As the exponent of the power weighting increases, the distribution undergoes a phase transition from the normal contribution being dominant to the boundary effects dominating. In the critical case where the weight is simple Euclidean length, both effects contribute significantly to the limit law. We also give a law of large numbers for the total weight of the graph.

Key words and phrases: Spanning tree; nearest neighbour graph; weak convergence; fixed-point equation; phase transition; fragmentation process.

1 Introduction

Recent interest in graphs, generated over random point sets consisting of independent uniform points in the unit square by connecting nearby points according to some deterministic rule, has been considerable. Such graphs include the geometric graph, the nearest neighbour graph and the minimal-length spanning tree. Many aspects of the large-sample asymptotic theory for such graphs, when they are locally determined in a certain sense, are by now quite well understood. See for example [9, 14, 17, 18, 23, 24, 25].

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One such graph is the *minimal directed spanning tree* (or MDST for short), which was introduced by Bhatt and Roy in [6]. In the MDST, each point \mathbf{x} of a finite (random) subset \mathcal{S} of $(0, 1]^2$ is connected by a directed edge to the nearest $\mathbf{y} \in \mathcal{S} \cup \{(0, 0)\}$ such that $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{y} \preceq^* \mathbf{x}$, where $\mathbf{y} \preceq^* \mathbf{x}$ means that each component of $\mathbf{x} - \mathbf{y}$ is nonnegative. See Figure 1 for a realisation of the MDST on simulated random points.

Motivation comes from the modelling of communications or drainage networks (see [6, 16, 20]). For example, consider the problem of designing a set of canals to connect a set of hubs, so as to minimize their total length subject to a constraint that all canals must flow downhill. The mathematical formulation given above for this constraint can lead to significant boundary effects due to the possibility of long edges occurring near the lower and left boundaries of the unit square; these boundary effects distinguish the MDST qualitatively from the standard minimal spanning tree and the nearest neighbour graph for point sets in the plane. Another difference is the fact that there is no uniform upper bound on vertex degrees in the MDST.

In the present work, we consider the total length of the MDST on random points in $(0, 1]^2$, as the number of points becomes large. We also consider the total length of the *minimal directed spanning forest* (MDSF), which is the MDST with edges incident to the origin removed (see Figure 1 for an example). In [6], Bhatt and Roy mention that the total length is an object of considerable interest, although they restrict their analysis to the length of the edges joined to the origin (subsequently also examined in [16]). A first order result for the total length of the MDST or MDSF is a law of large numbers; we derive this in Theorem 2.1 for a family of MDSFs indexed by partial orderings on \mathbf{R}^2 , which include \preceq^* as a special case.

This paper is mainly concerned with establishing second order results, i.e., weak convergence results for the distribution of the total length, suitably centred and scaled. For the length of edges from points in the region away from the boundary, we prove a central limit theorem. The boundary effects are significant, and near the boundary the MDST can be described in terms of a one-dimensional, on-line version of the MDST which we call the directed linear tree (DLT), and which we examine in Section 3. In the DLT, each point in a sequence of independent uniform random points in an interval is joined to its nearest neighbour to the left, amongst those points arriving earlier in the sequence. This DLT is of separate interest in relation to, for example, network modelling and molecular fragmentation (see [5], [4], and references therein).

In Theorem 3.1 we establish that the limiting distribution of the centred total length of the DLT is characterized by a distributional fixed-point equation, which resembles those encountered in the probabilistic analysis of algorithms such as Quick-sort [7]. Such fixed-point distributional equalities, and the so-called ‘divide and conquer’ or recursive algorithms from which they arise, have received considerable attention recently; see, for example, [8, 13, 21, 22].

We consider power-weighted edges. Our weak convergence results (Theorem 2.2) demonstrate that, depending on the value chosen for the weight exponent of the edges, there are two regimes in which either the boundary effects dominate

or those edges away from the boundary are dominant, and that there is a critical value (when we take simple Euclidean length as the weight) for which neither effect dominates.

In the related paper [16], we give results dealing with the weight of the edges joined to the origin, including weak convergence results, in which the limiting distributions are given in terms of some generalized Dickman distributions. Subsequently, it has been shown [2] that this two dimensional case is rather special – in higher dimensions the corresponding limits are normally distributed. [16] also deals with the maximum edge length of the MDST (the maximum length of those edges incident to the origin was dealt with in [6]).

In the next section we give formal definitions of the MDST and MDSF, and state our main results (Theorems 2.1 and 2.2) on the total length of the MDST and MDSF. The results on the DLT which we present in Section 3, and the general central limit theorems which we present in Section 4, are of some independent interest.

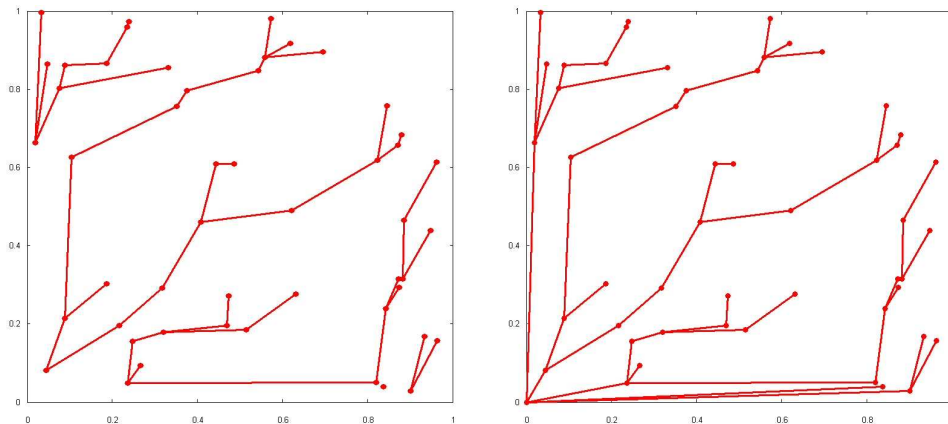


Figure 1: Realizations of the MDSF (left) and MDST on 100 simulated random points in the unit square, under the partial ordering \preceq^* .

2 Definitions and main results

We work in the same framework as [16]. Here we briefly recall the relevant terminology. See [16] for more detail.

Suppose V is a finite set endowed with a partial ordering \preceq . A *minimal element*, or *sink*, of V is a vertex $v_0 \in V$ for which there exists no $v \in V \setminus \{v_0\}$ such that $v \preceq v_0$. Let V_0 denote the set of all sinks of V .

The partial ordering induces a directed graph $G = (V, E)$, with vertex set V and with edge set E consisting of all ordered pairs (v, u) of distinct elements of V such that $u \preceq v$. A *directed spanning forest (DSF)* on V is a subgraph $T = (V_T, E_T)$ of (V, E) such that (i) $V_T = V$ and $E_T \subseteq E$, and (ii) for each vertex $v \in V \setminus V_0$ there exists a unique directed path in T that starts at v and ends at some sink $u \in V_0$. In

the case where V_0 consists of a single sink, we refer to any DSF on V as a *directed spanning tree (DST)* on V . If we ignore the orientation of edges then [16] a DSF on V is indeed a forest and, if there is just one sink, then any DST on V is a tree.

Suppose the directed graph (V, E) carries a *weight function* on its edges, i.e., a function $w : E \rightarrow [0, \infty)$. If T is a DSF on V , we set $w(T) := \sum_{e \in E_T} w(e)$. A *minimal directed spanning forest (MDSF)* on V (or, equivalently, on G), is a directed spanning forest T on V such that $w(T) \leq w(T')$ for every DSF T' on V . If V has a single sink, then a minimal directed spanning forest on V is called a *minimal directed spanning tree (MDST)* on V .

For $v \in V$, we say that $u \in V \setminus \{v\}$ is a *directed nearest neighbour* of v if $u \preceq v$ and $w(v, u) \leq w(v, u')$ for all $u' \in V \setminus \{v\}$ such that $u' \preceq v$. For each $v \in V \setminus V_0$, let n_v denote a directed nearest neighbour of v (chosen arbitrarily if v has more than one directed nearest neighbour). Then [16] the subgraph (V, E_M) of (V, E) , obtained by taking $E_M := \{(v, n_v) : v \in V \setminus V_0\}$, is a MDSF of V . Thus, if all edge-weights are distinct, the MDSF is unique, and is obtained by connecting each non-minimal vertex to its directed nearest neighbour.

For what follows, we consider a general type of partial ordering of \mathbf{R}^2 , denoted $\preceq_{\theta, \phi}$, specified by the angles $\theta \in [0, 2\pi)$ and $\phi \in (0, \pi] \cup \{2\pi\}$. For $\mathbf{x} \in \mathbf{R}^2$, let $C_{\theta, \phi}(\mathbf{x})$ be the closed cone with vertex \mathbf{x} and boundaries given by the rays from \mathbf{x} at angles θ and $\theta + \phi$, measuring anticlockwise from the upwards vertical. The partial order is such that, for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^2$,

$$\mathbf{x}_1 \preceq_{\theta, \phi} \mathbf{x}_2 \text{ iff } \mathbf{x}_1 \in C_{\theta, \phi}(\mathbf{x}_2). \quad (1)$$

We shall use \preceq^* as shorthand for the special case $\preceq_{\pi/2, \pi/2}$, which is of particular interest, as in [6]. In this case $u \preceq^* v$ for $u = (u_1, u_2), v = (v_1, v_2) \in E$ if and only if $u_1 \leq v_1$ and $u_2 \leq v_2$. The symbol \preceq will denote a general partial order on \mathbf{R}^2 .

We do not permit here the case $\phi = 0$, which would almost surely give us a disconnected point set. Nor do we allow $\pi < \phi < 2\pi$, since in this case the directional relation (1) is not a partial order, since the transitivity property (if $u \preceq v$ and $v \preceq w$ then $u \preceq w$) fails for $\pi < \phi < 2\pi$. We shall, however, allow the case $\phi = 2\pi$ which leads to the standard nearest neighbour (directed) graph.

The weight function is given by power-weighted Euclidean distance, i.e., for $(u, v) \in E$ we assign weight $w(u, v) = \|u - v\|^\alpha$ to the edge (u, v) , where $\|\cdot\|$ denotes the Euclidean norm on \mathbf{R}^2 , and $\alpha > 0$ is an arbitrary fixed parameter. Thus, when $\alpha = 1$ the weight of an edge is simply its Euclidean length. Moreover, we shall assume that $V \subset \mathbf{R}^2$ is given by $V = \mathcal{S}$ or $V = \mathcal{S}^0 := \mathcal{S} \cup \{\mathbf{0}\}$, where $\mathbf{0}$ is the origin in \mathbf{R}^2 and \mathcal{S} is generated in a *random* manner. The random point set \mathcal{S} will usually be either the set of points given by a homogeneous Poisson point process \mathcal{P}_n of intensity n on the unit square $(0, 1]^2$, or a binomial point process \mathcal{X}_n consisting of n independent uniformly distributed points on $(0, 1]^2$.

Note that in this random setting, each point of \mathcal{S} almost surely has a unique directed nearest neighbour, so that V has a unique MDSF, which does not depend

on the choice of α . Denote by $\mathcal{L}^\alpha(\mathcal{S})$ the total weight of all the edges in the MDSF on \mathcal{S} , and let $\tilde{\mathcal{L}}^\alpha(\mathcal{S}) := \mathcal{L}^\alpha(\mathcal{S}) - E[\mathcal{L}^\alpha(\mathcal{S})]$, the centred total weight.

Our first result presents laws of large numbers for the total edge weight for the general partial order $\preceq^{\theta, \phi}$ and general $0 < \alpha < 2$. We state the result for n points uniformly distributed on $(0, 1]^2$, but the proof carries through to other distributions (see the start of Section 5).

Theorem 2.1 *Suppose $0 < \alpha < 2$. Under the general partial order $\preceq^{\theta, \phi}$, with $0 \leq \theta < 2\pi$ and $0 < \phi \leq \pi$ or $\phi = 2\pi$, it is the case that*

$$n^{(\alpha/2)-1} \mathcal{L}^\alpha(\mathcal{X}_n) \xrightarrow{L^1} (2/\phi)^{\alpha/2} \Gamma(1 + \alpha/2), \quad \text{as } n \rightarrow \infty. \quad (2)$$

Also, when the partial order is \preceq^* , (2) remains true with the addition of the origin, i.e. with \mathcal{X}_n replaced by \mathcal{X}_n^0 .

Remark. In the special case $\alpha = 1$, the limit in (2) is $\sqrt{\pi/(2\phi)}$. This limit is 1 when $\phi = \pi/2$. Also, for $\phi = 2\pi$ we have the standard nearest neighbour (directed) graph (that is, every point is joined to its nearest neighbour by a directed edge), and this limit is then $1/2$. This result (for $\alpha = 1, \phi = 2\pi$) is stated without proof (and attributed to Miles [12]) in [1], but we have not previously seen the limiting constant derived explicitly, either in [12] or anywhere else.

Our main result (Theorem 2.2) presents convergence in distribution for the case where the partial order is \preceq^* ; the limiting distributions are of a different type in the three cases $\alpha = 1$ (the same situation as [6]), $0 < \alpha < 1$, and $\alpha > 1$. We define these limiting distributions in Theorem 2.2, in terms of distributional fixed-point equations. These fixed-point equations are of the form

$$X \stackrel{\mathcal{D}}{=} \sum_{r=1}^k A_r X^{\{r\}} + B, \quad (3)$$

where $k \in \mathbb{N}$, $X^{\{r\}}$, $r = 1, \dots, k$, are independent copies of the random variable X , and (A_1, \dots, A_k, B) is a random vector, independent of $(X^{\{1\}}, \dots, X^{\{k\}})$, satisfying the conditions

$$E \sum_{r=1}^k |A_r|^2 < 1, \quad E[B] = 0, \quad E[B^2] < \infty. \quad (4)$$

Theorem 3 of Rösler [21] (proved using the contraction mapping theorem; see also [13, 22]) says that if (4) holds, there is a unique square-integrable distribution with mean zero satisfying the fixed-point equation (3), and this will guarantee uniqueness of solutions to all the distributional fixed-point equalities considered in the sequel.

Define the random variable \tilde{D}_1 , to have the distribution that is the unique solution to the distributional fixed-point equation

$$\tilde{D}_1 \stackrel{\mathcal{D}}{=} U \tilde{D}_1^{\{1\}} + (1 - U) \tilde{D}_1^{\{2\}} + U \log U + (1 - U) \log(1 - U) + U, \quad (5)$$

where U is uniform on $(0, 1)$ and independent of the other variables on the right. We shall see later (in Propositions 3.5 and 3.6) that $E[\tilde{D}_1] = 0$ and $\text{Var}[\tilde{D}_1] = 2 - \pi^2/6$; higher order moments are given recursively by eqn (14).

For $\alpha > 1$, let \tilde{D}_α denote a random variable with distribution characterized by the fixed-point equation

$$\tilde{D}_\alpha \stackrel{\mathcal{D}}{=} U^\alpha \tilde{D}_\alpha^{\{1\}} + (1 - U)^\alpha \tilde{D}_\alpha^{\{2\}} + \frac{\alpha}{\alpha - 1} U^\alpha + \frac{1}{\alpha - 1} (1 - U)^\alpha - \frac{1}{\alpha - 1}, \quad (6)$$

where again U is uniform on $(0, 1)$ and independent of the other variables on the right. Also for $\alpha > 1$, let \tilde{F}_α denote a random variable with distribution characterized by the fixed-point equation

$$\tilde{F}_\alpha \stackrel{\mathcal{D}}{=} U^\alpha \tilde{F}_\alpha + (1 - U)^\alpha \tilde{D}_\alpha + \frac{U^\alpha}{\alpha(\alpha - 1)} + \frac{(1 - U)^\alpha}{\alpha - 1} - \frac{1}{\alpha(\alpha - 1)}, \quad (7)$$

where U is uniform on $(0, 1)$, \tilde{D}_α has the distribution given by (6), and the U , \tilde{D}_α and \tilde{F}_α on the right are independent. In Section 3 we shall see that the random variables \tilde{D}_α , \tilde{F}_α for $\alpha > 1$ arise as centred versions of random variables (denoted D_α , F_α respectively) satisfying somewhat simpler fixed point equations. Thus \tilde{D}_α and \tilde{F}_α both have mean zero; their variances are given by eqns (38) and (40) below.

Let $\mathcal{N}(0, s^2)$ denote the normal distribution with mean zero and variance s^2 .

Theorem 2.2 *Suppose the weight exponent is $\alpha > 0$ and the partial order is \preceq^* . There exist constants $0 < t_\alpha^2 \leq s_\alpha^2$ such that, for normal random variables $Y_\alpha \sim \mathcal{N}(0, s_\alpha^2)$ and $W_\alpha \sim \mathcal{N}(0, t_\alpha^2)$:*

(i) *As $n \rightarrow \infty$,*

$$n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n^0) \xrightarrow{\mathcal{D}} Y_\alpha \quad \text{and} \quad n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n^0) \xrightarrow{\mathcal{D}} W_\alpha \quad (0 < \alpha < 1); \quad (8)$$

$$\tilde{\mathcal{L}}^1(\mathcal{P}_n^0) \xrightarrow{\mathcal{D}} \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}} + Y_1 \quad \text{and} \quad \tilde{\mathcal{L}}^1(\mathcal{X}_n^0) \xrightarrow{\mathcal{D}} \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}} + W_1; \quad (9)$$

$$\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n^0) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha^{\{1\}} + \tilde{D}_\alpha^{\{2\}} \quad \text{and} \quad \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n^0) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha^{\{1\}} + \tilde{D}_\alpha^{\{2\}} \quad (\alpha > 1). \quad (10)$$

Here all the random variables in the limits are independent, and $\tilde{D}_\alpha^{\{i\}}$, $i = 1, 2$ are independent copies of the random variable \tilde{D}_α defined at (5) for $\alpha = 1$ and (6) for $\alpha > 1$.

(ii) *As $n \rightarrow \infty$,*

$$n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n) \xrightarrow{\mathcal{D}} Y_\alpha \quad \text{and} \quad n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n) \xrightarrow{\mathcal{D}} W_\alpha \quad (0 < \alpha < 1); \quad (11)$$

$$\tilde{\mathcal{L}}^1(\mathcal{P}_n) \xrightarrow{\mathcal{D}} \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}} + Y_1 \quad \text{and} \quad \tilde{\mathcal{L}}^1(\mathcal{X}_n) \xrightarrow{\mathcal{D}} \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}} + W_1; \quad (12)$$

$$\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}} \quad \text{and} \quad \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}} \quad (\alpha > 1). \quad (13)$$

Here all the random variables in the limits are independent, and $\tilde{D}_1^{\{i\}}$, $i = 1, 2$, are independent copies of \tilde{D}_1 with distribution defined at (5), and for $\alpha > 1$, $\tilde{F}_\alpha^{\{i\}}$, $i = 1, 2$, are independent copies of \tilde{F}_α with distribution defined at (7).

Remarks. The normal random variables Y_α or W_α arise from the edges away from the boundary (see Section 6). The non-normal variables (the \tilde{D} s and \tilde{F} s) arise from the edges very close to the boundary, where the MDSF is asymptotically close to the ‘directed linear forest’ discussed in Section 3.

Theorem 2.2 indicates a phase transition in the character of the limit law as α increases. The normal contribution (from the points away from the boundary) dominates for $0 < \alpha < 1$, while the boundary contributions dominate for $\alpha > 1$. In the critical case $\alpha = 1$, neither effect dominates and both terms contribute significantly to the asymptotic behaviour.

Noteworthy in the case $\alpha = 1$ is the fact that by (9) and (12), the limiting distribution is the same for $\tilde{\mathcal{L}}^1(\mathcal{P}_n)$ as for $\tilde{\mathcal{L}}^1(\mathcal{P}_n^0)$, and the same for $\tilde{\mathcal{L}}^1(\mathcal{X}_n)$ as for $\tilde{\mathcal{L}}^1(\mathcal{X}_n^0)$. Note, however, that the difference $\tilde{\mathcal{L}}^1(\mathcal{P}_n) - \tilde{\mathcal{L}}^1(\mathcal{P}_n^0)$ is the (centred) total length of edges incident to the origin, which is not negligible, but itself converges in distribution (see [16]) to a non-degenerate random variable, namely a centred generalized Dickman random variable with parameter 2 (see (28) below). As an extension of Theorem 2.2, it should be possible to show that the joint distribution of $(\tilde{\mathcal{L}}^1(\mathcal{P}_n), \tilde{\mathcal{L}}^1(\mathcal{P}_n^0))$ converges to that of two coupled random variables, both having the distribution of \tilde{D}_1 , whose difference has the centred generalized Dickman distribution with parameter 2. Likewise for the joint distribution of $(\tilde{\mathcal{L}}^1(\mathcal{X}_n), \tilde{\mathcal{L}}^1(\mathcal{X}_n^0))$.

Of particular interest is the distribution of the variable \tilde{D}_1 appearing in Theorem 2.2. In Section 3.4, we give a plot (Figure 2) of the probability density function of this distribution, estimated by simulation. Also, we can use the fixed-point equation (5) to calculate the moments of \tilde{D}_1 recursively. Writing

$$f(U) := U \log U + (1 - U) \log(1 - U) + U,$$

and setting $m_k := E[\tilde{D}_1^k]$, we obtain

$$m_k = E[(f(U))^k] + \sum_{i=2}^k \binom{k}{i} \sum_{j=0}^i \binom{i}{j} E[(f(U))^{k-i} U^j (1 - U)^{i-j}] m_j m_{i-j}. \quad (14)$$

The fact that $m_1 = 0$ simplifies things a little, and we can rewrite this as

$$m_k = E[(f(U))^k] + \sum_{i=1}^k \binom{k}{i} \left[m_i E[(f(U))^{k-i} (U^i + (1 - U)^i)] + \sum_{j=2}^{i-2} \binom{i}{j} E[(f(U))^{k-i} U^j (1 - U)^{i-j}] m_j m_{i-j} \right].$$

So, for example, when $k = 3$ we obtain $m_3 \approx 0.15411$, which shows \tilde{D}_1 is not Gaussian and is consistent with the skewness of the plot in Figure 2.

The remainder of this paper is organized as follows. After discussion of the DLT in Section 3, in Section 4 we present general limit theorems in geometric probability, which we shall use in obtaining our main results for the MDST. Theorem 2.1 is

proved in Section 5 (this proof does not use the results of Section 3). The proof of Theorem 2.2 is prepared in Sections 6 and 7, and completed in Section 8. In these proofs, we repeatedly use *Slutsky's theorem* (see e.g. [14]) which says that if $X_n \rightarrow X$ in distribution and $Y_n \rightarrow 0$ in probability, then $X_n + Y_n \rightarrow X$ in distribution.

3 The directed linear forest and tree

The directed linear forest (DLF) and directed linear tree (DLT) are for us a tool for the analysis of the limiting behaviour of the contribution to the total weight of the random MDSF/MDST from edges near the boundary of the unit square. In the present section we derive the properties of the DLF that we need (in particular, Theorem 3.1); subsequently, in Theorem 7.1, we shall see that the total weight of edges from the points near the boundaries, as $n \rightarrow \infty$, converges in distribution to the limit of the total weight of the DLF.

The DLT is also of some intrinsic interest. It is a one-dimensional directed analogue of the so-called ‘on-line nearest neighbour graph’, which is of interest in the study of networks such as the world wide web (see, e.g. [5]; and [15] for more on the on-line nearest neighbour graph). Moreover, it is constructed via a fragmentation process similar to those seen in, for example, [4]; the tree provides a historical representation of the fragmentation process.

For any finite sequence $\mathcal{T}_m = (x_1, x_2, \dots, x_m) \in (0, 1]^m$, we construct the directed linear forest (DLF) as follows. We start with the unit interval $(0, 1]$ and insert the points x_i in order, one at a time, starting with $i = 1$. At the insertion of each point, we join the new point to its nearest neighbour among those points already present that lie to the *left* of the point (provided that such a point exists). In other words, for each point x_i , $i \geq 2$, we join x_i by a directed edge to the point $\max\{x_j : 1 \leq j < i, x_j < x_i\}$. If $\{x_j : 1 \leq j < i, x_j < x_i\}$ is empty, we do not add any directed edge from x_i . In this way we construct a ‘directed linear forest’, which we denote by $\text{DLF}(\mathcal{T}_m)$. We denote the total weight (under weight function with exponent α) of $\text{DLF}(\mathcal{T}_m)$ by $D^\alpha(\mathcal{T}_m)$, that is, we set

$$D^\alpha(\mathcal{T}_m) := \sum_{i=2}^m (x_i - \max\{x_j : 1 \leq j < i, x_j < x_i\})^\alpha \mathbf{1}_{\{\min\{x_j : 1 \leq j < i\} < x_i\}}.$$

Further, given \mathcal{T}_m , let \mathcal{T}_m^0 be the sequence (x_0, x_1, \dots, x_m) where the initial term is $x_0 := 0$. Then the DLT on \mathcal{T}_m^0 is constructed in the same way, where now for each $i \geq 1$, we join x_i by an edge to the point $\max\{x_j : 0 \leq j < i, x_j < x_i\}$. But now we see that x_1 will always be joined to $x_0 = 0$, and x_2 will be joined either to x_1 (if $x_2 > x_1$) or to x_0 , and so on. In this way we construct a ‘directed linear tree’ (DLT) on vertex set $\{x_0, x_1, \dots, x_m\}$ with m edges. Denote the total weight of this tree with weight exponent α by $D^\alpha(\mathcal{T}_m^0)$; that is, set

$$D^\alpha(\mathcal{T}_m^0) := \sum_{i=1}^m (x_i - \max\{x_j : 0 \leq j < i, x_j < x_i\})^\alpha.$$

We shall be mainly interested in the case where \mathcal{T}_m is a random vector in $(0, 1]^m$. In this case, set $\tilde{D}^\alpha(\mathcal{T}_m) := D^\alpha(\mathcal{T}_m) - E[D^\alpha(\mathcal{T}_m)]$ the centred total weight of the DLF, and $\tilde{D}^\alpha(\mathcal{T}_m^0) = D^\alpha(\mathcal{T}_m^0) - E[D^\alpha(\mathcal{T}_m^0)]$ the centred total weight of the DLT.

We take \mathcal{T}_m to be a vector of uniform variables. Let (X_1, X_2, X_3, \dots) be a sequence of independent uniformly distributed random variables in $(0, 1]$, and for $m \in \mathbf{N}$ set $\mathcal{U}_m := (X_1, X_2, \dots, X_m)$. We consider $D^\alpha(\mathcal{U}_m)$ and $D^\alpha(\mathcal{U}_m^0)$. For these variables, we establish asymptotic behaviour of the mean value in Propositions 3.1 and 3.2, along with the following convergence results, which are the principal results of this section.

For $\alpha > 1$, let D_α denote a random variable with distribution characterized by the fixed-point equation

$$D_\alpha \stackrel{\mathcal{D}}{=} U^\alpha D_\alpha^{\{1\}} + (1 - U)^\alpha D_\alpha^{\{2\}} + U^\alpha, \quad (15)$$

where U is uniform on $(0, 1)$ and independent of the other variables on the right. Also for $\alpha > 1$, let F_α denote a random variable with distribution characterized by the fixed-point equation

$$F_\alpha \stackrel{\mathcal{D}}{=} U^\alpha F_\alpha + (1 - U)^\alpha D_\alpha, \quad (16)$$

where U is uniform on $(0, 1)$, D_α has the distribution given by (15), and the U , D_α and F_α on the right are independent. The corresponding centred random variables $\tilde{D}_\alpha := D_\alpha - E[D_\alpha]$ and $\tilde{F}_\alpha := F_\alpha - E[F_\alpha]$ satisfy the fixed-point equations (6) and (7) respectively. The solutions to (6) and (7) are unique by the criterion given at (4), and hence the solutions to (15) and (16) are also unique.

Theorem 3.1 (i) As $m \rightarrow \infty$ we have $\tilde{D}^1(\mathcal{U}_m^0) \xrightarrow{L^2} \tilde{D}_1$ and $\tilde{D}^1(\mathcal{U}_m) \xrightarrow{L^2} \tilde{F}_1$ where \tilde{D}_1 has the distribution given by the fixed-point equation (5), and \tilde{F}_1 has the same distribution as \tilde{D}_1 . Also, the variance of \tilde{D}_1 (and hence also of \tilde{F}_1) is $2 - \pi^2/6 \approx 0.355066$. Finally, $\text{Cov}(\tilde{D}_1, \tilde{F}_1) = (7/4) - \pi^2/6 \approx 0.105066$.

(ii) For $\alpha > 1$, as $m \rightarrow \infty$ we have $D^\alpha(\mathcal{U}_m^0) \rightarrow D_\alpha$, almost surely and in L^2 , and $D^\alpha(\mathcal{U}_m) \xrightarrow{L^2} F_\alpha$, almost surely and in L^2 , where the distributions of D_α , F_α are given by the fixed-point equations (15) and (16) respectively. Also, $E[D_\alpha] = (\alpha - 1)^{-1}$ and $E[F_\alpha] = (\alpha(\alpha - 1))^{-1}$, while $\text{Var}(D_\alpha)$ and $\text{Var}(F_\alpha)$ are given by (38) and (40) respectively.

Proof. Part (i) follows from Propositions 3.5, 3.6 and 3.7 below. Part (ii) follows from Propositions 3.3 and 3.4 below. We prove these results in the following sections. \square

An interesting property of the DLT, which we use in establishing fixed-point equations for limit distributions, is its *self-similarity* (scaling property). In terms of the total weight, this says that for any $t \in (0, 1)$, if Y_1, \dots, Y_n are independent and uniformly distributed on $(0, t]$, then the distribution of $D^\alpha(Y_1, \dots, Y_n)$ is the same as that of $t^\alpha D^\alpha(X_1, \dots, X_n)$.

3.1 The mean total weight of the DLF and DLT

First we consider the rooted case, i.e. the DLT on \mathcal{U}_m^0 . For $m = 1, 2, 3, \dots$ denote by Z_m the random variable given by the gain in length of the tree on the addition of one point (X_m) to an existing $m - 1$ points in the DLT on a sequence of uniform random variables \mathcal{U}_{m-1}^0 , i.e. with the conventions $D^1(\mathcal{U}_0^0) = 0$ and $X_0 = 0$, we set

$$Z_m := D^1(\mathcal{U}_m^0) - D^1(\mathcal{U}_{m-1}^0) = X_m - \max\{X_j : 0 \leq j < m, X_j < X_m\}. \quad (17)$$

Thus, with weight exponent α , the m th edge to be added has weight Z_m^α .

Lemma 3.1 (i) Z_m has distribution function F_m given by $F_m(t) = 0$ for $t < 0$, $F_m(t) = 1$ for $t > 1$, and $F_m(t) = 1 - (1 - t)^m$ for $0 \leq t \leq 1$.

(ii) For $\alpha > 0$, Z_m^α has expectation and variance

$$E[Z_m^\alpha] = \frac{m!\Gamma(1 + \alpha)}{\Gamma(1 + \alpha + m)}, \quad \text{Var}[Z_m^\alpha] = \frac{m!\Gamma(1 + 2\alpha)}{\Gamma(1 + 2\alpha + m)} - \left(\frac{m!\Gamma(1 + \alpha)}{\Gamma(1 + \alpha + m)} \right)^2. \quad (18)$$

In particular,

$$E[Z_m] = \frac{1}{m + 1}; \quad \text{Var}[Z_m] = \frac{m}{(m + 1)^2(m + 2)}. \quad (19)$$

(iii) For $\alpha > 0$, as $m \rightarrow \infty$ we have

$$E[Z_m^\alpha] \sim \Gamma(\alpha + 1)m^{-\alpha}, \quad \text{Var}[Z_m^\alpha] \sim (\Gamma(2\alpha + 1) - (\Gamma(\alpha + 1))^2) m^{-2\alpha}. \quad (20)$$

(iv) As $m \rightarrow \infty$, mZ_m converges in distribution, to an exponential with parameter 1.

Proof. For $0 \leq t \leq 1$ we have

$$P[Z_m > t] = P[X_m > t \text{ and none of } X_1, \dots, X_{m-1} \text{ lies in } (X_m - t, X_m)] = (1 - t)^m,$$

and (i) follows. We then obtain (ii) since for any $\alpha > 0$ and for $k = 1, 2$,

$$E[Z_m^{k\alpha}] = \int_0^1 P[Z_m > t^{1/(k\alpha)}] dt = \int_0^1 (1 - t^{1/k\alpha})^m dt = \frac{m!\Gamma(1 + k\alpha)}{\Gamma(1 + k\alpha + m)}.$$

Then (iii) follows by Stirling's formula, which yields

$$E[Z_m^{k\alpha}] = \Gamma(1 + k\alpha)m^{-k\alpha}(1 + O(m^{-1})).$$

For (iv), we have from (i) that, for $t \in [0, \infty)$, and m large enough so that $(t/m) \leq 1$,

$$P[mZ_m \leq t] = F_m\left(\frac{t}{m}\right) = 1 - \left(1 - \frac{t}{m}\right)^m \rightarrow 1 - e^{-t}, \text{ as } m \rightarrow \infty.$$

But $1 - e^{-t}$, $t \geq 0$ is the exponential distribution function with parameter 1. \square

The following result gives the asymptotic behaviour of the expected total weight of the DLT. Let γ denote Euler's constant, so that

$$\left(\sum_{i=1}^k \frac{1}{i} \right) - \log k = \gamma + O(k^{-1}). \quad (21)$$

Proposition 3.1 *As $m \rightarrow \infty$ the expected total weight of the DLT under α -power weighting on \mathcal{U}_m^0 satisfies*

$$E[D^\alpha(\mathcal{U}_m^0)] \sim \frac{\Gamma(\alpha+1)}{1-\alpha} m^{1-\alpha} \quad (0 < \alpha < 1); \quad (22)$$

$$E[D^1(\mathcal{U}_m^0)] - \log m \rightarrow \gamma - 1; \quad (23)$$

$$E[D^\alpha(\mathcal{U}_m^0)] = \frac{1}{\alpha-1} + O(m^{1-\alpha}) \quad (\alpha > 1). \quad (24)$$

Proof. We have

$$E[D^\alpha(\mathcal{U}_m^0)] = \sum_{i=1}^m (E[D^\alpha(\mathcal{U}_i^0)] - E[D^\alpha(\mathcal{U}_{i-1}^0)]) = \sum_{i=1}^m E[Z_i^\alpha].$$

In the case where $\alpha = 1$, $E[Z_i] = (i+1)^{-1}$ by (19), and (23) follows by (21). For general $\alpha > 0$, $\alpha \neq 1$, from (18) we have that

$$E[D^\alpha(\mathcal{U}_m^0)] = \Gamma(1+\alpha) \sum_{i=1}^m \frac{\Gamma(i+1)}{\Gamma(1+\alpha+i)} = \frac{1}{\alpha-1} - \frac{\Gamma(1+\alpha)\Gamma(m+2)}{(\alpha-1)\Gamma(m+1+\alpha)}. \quad (25)$$

By Stirling's formula, the last term satisfies

$$- \frac{\Gamma(1+\alpha)\Gamma(m+2)}{(\alpha-1)\Gamma(m+1+\alpha)} = - \frac{\Gamma(1+\alpha)}{\alpha-1} m^{1-\alpha} (1 + O(m^{-1})), \quad (26)$$

which tends to zero as $m \rightarrow \infty$ for $\alpha > 1$, to give us (24). For $\alpha < 1$, we have (22) from (25) and (26). \square

Now consider the unrooted case, i.e., the directed linear forest. For \mathcal{U}_m as above the total weight of the DLF is denoted $D^\alpha(\mathcal{U}_m)$, and the centred total weight is $\tilde{D}^\alpha(\mathcal{U}_m) := D^\alpha(\mathcal{U}_m) - E[D^\alpha(\mathcal{U}_m)]$. We then see that

$$D^\alpha(\mathcal{U}_m^0) = D^\alpha(\mathcal{U}_m) + \mathcal{L}_0^\alpha(\mathcal{U}_m^0), \quad (27)$$

where $\mathcal{L}_0^\alpha(\mathcal{U}_m^0)$ is the total weight of edges incident to 0 in the DLT on \mathcal{U}_m^0 .

The following lemma says that $\mathcal{L}_0^\alpha(\mathcal{U}_m^0)$ converges to a random variable that has the generalized Dickman distribution with parameter $1/\alpha$ (see [16]), that is, the distribution of a random variable X which satisfies the distributional fixed-point equation

$$X \stackrel{\mathcal{D}}{=} U^\alpha(1+X), \quad (28)$$

where U is uniform on $(0, 1)$ and independent of the X on the right. We recall from Proposition 3 of [16] that if X satisfies (28) then

$$E[X] = 1/\alpha, \text{ and } E[X^2] = (\alpha+2)/(2\alpha^2). \quad (29)$$

Lemma 3.2 *Let $\alpha > 0$. There is a random variable \mathcal{L}_0^α with the generalized Dickman distribution with parameter $1/\alpha$, such that as $m \rightarrow \infty$, we have that $\mathcal{L}_0^\alpha(\mathcal{U}_m^0) \rightarrow \mathcal{L}_0^\alpha$, almost surely and in L^2 .*

Proof. Let $\delta_D(\mathcal{U}_m^0)$ denote the degree of the origin in the directed linear tree on \mathcal{U}_m^0 , so that $\delta_D(\mathcal{U}_m^0)$ is the number of lower records in the sequence (X_1, \dots, X_m) . Then

$$\mathcal{L}_0^\alpha(\mathcal{U}_m^0) = U_1^\alpha + (U_1 U_2)^\alpha + \dots + (U_1 \dots U_{\delta_D(\mathcal{U}_m^0)})^\alpha, \quad (30)$$

where (U_1, U_2, \dots) is a certain sequence of independent uniform random variables on $(0, 1)$, namely the ratios between successive lower records of the sequence (X_n) . The sum $U_1^\alpha + (U_1 U_2)^\alpha + (U_1 U_2 U_3)^\alpha + \dots$ has nonnegative terms and finite expectation, so it converges almost surely to a limit which we denote \mathcal{L}_0^α . Then \mathcal{L}_0^α has the generalized Dickman distribution with parameter $1/\alpha$ (see Proposition 2 of [16]).

Since $\delta_D(\mathcal{U}_m^0)$ tends to infinity almost surely as $m \rightarrow \infty$, we have $\mathcal{L}_0^\alpha(\mathcal{U}_m^0) \rightarrow \mathcal{L}_0^\alpha$ almost surely. Also, $E[(\mathcal{L}_0^\alpha)^2] < \infty$, by (29), and $(\mathcal{L}_0^\alpha - \mathcal{L}_0^\alpha(\mathcal{U}_m^0))^2 \leq (\mathcal{L}_0^\alpha)^2$ for all m . Thus $E[(\mathcal{L}_0^\alpha(\mathcal{U}_m^0) - \mathcal{L}_0^\alpha)^2] \rightarrow 0$ by the dominated convergence theorem, and so we have the L^2 convergence as well. \square

Proposition 3.2 *As $m \rightarrow \infty$ the expected total weight of the DLF under α -power weighting on \mathcal{U}_m satisfies*

$$E[D^\alpha(\mathcal{U}_m)] \sim \frac{\Gamma(\alpha + 1)}{1 - \alpha} m^{1-\alpha} \quad (0 < \alpha < 1); \quad (31)$$

$$E[D^1(\mathcal{U}_m)] - \log m \rightarrow \gamma - 2; \quad (32)$$

$$E[D^\alpha(\mathcal{U}_m)] \rightarrow \frac{1}{\alpha(\alpha - 1)} \quad (\alpha > 1). \quad (33)$$

Proof. By (27) we have $E[D^\alpha(\mathcal{U}_m)] = E[D^\alpha(\mathcal{U}_m^0)] - E[\mathcal{L}_0^\alpha(\mathcal{U}_m^0)]$. By Lemma 3.2 and (29),

$$E[\mathcal{L}_0^\alpha(\mathcal{U}_m^0)] \rightarrow E[\mathcal{L}_0^\alpha] = 1/\alpha.$$

We then obtain (31), (32) and (33) from Proposition 3.1. \square

3.2 Orthogonal increments for $\alpha = 1$

In this section we shall show (in Lemma 3.5) that when $\alpha = 1$, the variables $Z_i, i \geq 1$ are mutually orthogonal, in the sense of having zero covariances, which will be used later on to establish convergence of the (centred) total length of the DLT. To prove this, we first need further notation.

Given X_1, \dots, X_m , let us denote the order statistics of X_1, \dots, X_m , taken in increasing order, as $X_{(1)}^m, X_{(2)}^m, \dots, X_{(m)}^m$. Thus $(X_{(1)}^m, X_{(2)}^m, \dots, X_{(m)}^m)$ is a nondecreasing sequence, forming a permutation of the original (X_1, \dots, X_m) . Denote the existing $m + 1$ intervals

between points by $I_j^m := (X_{(j-1)}^m, X_{(j)}^m)$ for $j = 1, 2, \dots, m+1$, where we set $X_{(0)}^m := 0$ and $X_{(m+1)}^m := 1$. Let the widths of these intervals (the spacings) be

$$S_j^m := |I_j^m| = X_{(j)}^m - X_{(j-1)}^m,$$

for $1 \leq j \leq m+1$. Then $0 \leq S_j^m < 1$ for $1 \leq j \leq m+1$, and $\sum_{j=1}^{m+1} S_j^m = 1$. That is, the vector $(S_1^m, S_2^m, \dots, S_{m+1}^m)$ belongs to the m -dimensional simplex, Δ_m . Note that only m of the S_j^m are required to specify the vector.

We can arrange the spacings themselves $(S_j^m, 1 \leq j \leq m+1)$ into increasing order to give $S_{(1)}^m, S_{(2)}^m, \dots, S_{(m+1)}^m$. Then let \mathcal{F}_S^m denote the sigma field generated by these ordered spacings, so that

$$\mathcal{F}_S^m = \sigma(S_{(1)}^m, \dots, S_{(m+1)}^m). \quad (34)$$

The following interpretation of \mathcal{F}_S^m may be helpful. The set $(0, 1) \setminus \{X_1, \dots, X_m\}$ consists almost surely of $m+1$ connected components ('fragments') of total length 1, and \mathcal{F}_S^m is the σ -field generated by the collection of lengths of these fragments, ignoring the order in which they appear.

By definition, the value of Z_m must be one of the (ordered) spacings $S_{(1)}^m, \dots, S_{(m+1)}^m$. The next result says that, given the values of these spacings, each of the possible values for Z_m are equally likely.

Lemma 3.3 *For $m \geq 1$ we have*

$$P[Z_m = S_{(i)}^m | \mathcal{F}_S^m] = \frac{1}{m+1} \quad \text{a.s., for } i = 1, \dots, m+1. \quad (35)$$

Hence,

$$E[Z_m | \mathcal{F}_S^m] = \frac{1}{m+1} \sum_{i=1}^{m+1} S_{(i)}^m = \frac{1}{m+1}. \quad (36)$$

Proof. First we note that $(X_{(1)}^m, \dots, X_{(m)}^m)$ is uniformly distributed over

$$\{(x_1, \dots, x_m) : 0 \leq x_1 \leq x_2 \leq \dots \leq x_m \leq 1\}.$$

Now

$$\begin{pmatrix} S_1^m \\ S_2^m \\ S_3^m \\ \vdots \\ S_m^m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} X_{(1)}^m \\ X_{(2)}^m \\ X_{(3)}^m \\ \vdots \\ X_{(m)}^m \end{pmatrix}.$$

The m by m matrix here has determinant 1. Hence (S_1^m, \dots, S_m^m) is uniform over

$$\left\{ (x_1, \dots, x_m) : \sum_{j=1}^m x_j \leq 1; x_j \geq 0, \forall 1 \leq j \leq m \right\}.$$

Then $(S_1^m, \dots, S_{m+1}^m)$ is uniform over the m -dimensional simplex Δ_m . In particular, the S_j^m are exchangeable. Thus given $S_{(1)}^m, \dots, S_{(m+1)}^m$, i.e. \mathcal{F}_S^m , the actual values of S_1^m, \dots, S_{m+1}^m are equally likely to be any permutation of $S_{(1)}^m, \dots, S_{(m+1)}^m$, and given S_1^m, \dots, S_{m+1}^m the value of Z_m is equally likely to be any of S_1^m, \dots, S_m^m (but cannot be S_{m+1}^m).

Hence, given $S_{(1)}^m, \dots, S_{(m+1)}^m$ the probability that $Z_m = S_{(i)}^m$ is $(1/m) \times m/(m+1) = 1/(m+1)$, i.e. we have (35), and then (36) follows since $\sum_{j=1}^{m+1} S_{(j)}^m = 1$. \square

Lemma 3.4 *Let $1 \leq m < \ell$. Given \mathcal{F}_S^m , Z_ℓ and Z_m are conditionally independent.*

Proof. Given \mathcal{F}_S^m , we have $S_{(1)}^m, \dots, S_{(m+1)}^m$, and by (35), the (conditional) distribution of Z_m is uniform on $\{S_{(1)}^m, \dots, S_{(m+1)}^m\}$. The conditional distribution of Z_ℓ , $\ell > m$, given \mathcal{F}_S^m , depends only on $S_{(1)}^m, \dots, S_{(m+1)}^m$ and not which one of them Z_m happens to be. Hence Z_m and Z_ℓ are conditionally independent. \square

Lemma 3.5 *For $1 \leq m < \ell$, the random variables Z_m, Z_ℓ satisfy $\text{Cov}[Z_m, Z_\ell] = 0$.*

Proof. From Lemmas 3.4 and 3.3,

$$E[Z_m Z_\ell | \mathcal{F}_S^m] = E[Z_m | \mathcal{F}_S^m] E[Z_\ell | \mathcal{F}_S^m] = \frac{1}{m+1} E[Z_\ell | \mathcal{F}_S^m],$$

and by taking expectations we obtain

$$E[Z_m Z_\ell] = \frac{1}{m+1} E[Z_\ell] = \frac{1}{m+1} \cdot \frac{1}{\ell+1} = E[Z_m] \cdot E[Z_\ell].$$

Hence the covariance of Z_m and Z_ℓ is zero. \square

Remarks. (i) Calculations yield, for example, that $E[D^1(\mathcal{U}_1^0)] = E[Z_1] = 1/2$, $E[D^1(\mathcal{U}_2^0)] = 5/6$, and $\text{Var}[Z_1] = 1/12$, $\text{Var}[Z_2] = 1/18$, $\text{Var}[D^1(\mathcal{U}_2^0)] = 5/36$.

(ii) The orthogonality structure of the Z_m^α is unique to the $\alpha = 1$ case. For example, it can be shown that, for $\alpha > 0$,

$$E[Z_1^\alpha] E[Z_2^\alpha] = \frac{2}{(1+\alpha)^2(2+\alpha)}, \text{ and } E[Z_1^\alpha Z_2^\alpha] = \frac{1}{2(1+\alpha)^2} \left(1 + \frac{2\Gamma(\alpha+2)^2}{\Gamma(2\alpha+3)} \right).$$

Then

$$\text{Cov}[Z_1^\alpha, Z_2^\alpha] = \frac{(\alpha-2)\Gamma(2\alpha+3) + 2(\alpha+2)\Gamma(\alpha+2)^2}{2(\alpha+1)^2(\alpha+2)\Gamma(2\alpha+3)},$$

and this quantity is zero only if $\alpha = 1$; it is positive for $\alpha > 1$ and negative for $0 < \alpha < 1$.

3.3 Limit behaviour for $\alpha > 1$

We now consider the limit distribution of the total weight of the DLT and DLF. In the present section we consider the case of α -power weighted edges with $\alpha > 1$; that is, we prove part (ii) of Theorem 3.1. To describe the moments of the limiting distribution of $D^\alpha(\mathcal{U}_m^0)$ and $D^\alpha(\mathcal{U}_m)$, we introduce the notation

$$J(\alpha) := \int_0^1 u^\alpha (1-u)^\alpha du = 2^{-1-2\alpha} \sqrt{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}. \quad (37)$$

We start with the rooted case ($D^\alpha(\mathcal{U}_m^0)$), and subsequently consider the unrooted case ($D^\alpha(\mathcal{U}_m)$).

Proposition 3.3 *Let $\alpha > 1$. Then there exists a random variable D_α such that as $m \rightarrow \infty$ we have $D^\alpha(\mathcal{U}_m^0) \rightarrow D_\alpha$ almost surely and in L^2 . Also, the random variable D_α satisfies the distributional fixed-point equality (15). Further, $E[D_\alpha] = 1/(\alpha-1)$ and*

$$\text{Var}[D_\alpha] = \frac{\alpha(\alpha-2+2(2\alpha+1)J(\alpha))}{(\alpha-1)^2(2\alpha-1)}. \quad (38)$$

Proof. Let Z_i be the length of the i th edge of the DLT, as defined at (17). Let $D_\alpha := \sum_{i=1}^\infty Z_i^\alpha$. The sum converges almost surely since it has non-negative terms and, by (20), has finite expectation for $\alpha > 1$. By (20) and Cauchy-Schwarz, there exists a constant $0 < C < \infty$ such that

$$E[D_\alpha^2] = \sum_{i=1}^\infty \sum_{j=1}^\infty E[Z_i^\alpha Z_j^\alpha] \leq C \sum_{i=1}^\infty \sum_{j=1}^\infty i^{-\alpha} j^{-\alpha} < \infty,$$

since $\alpha > 1$. The L^2 convergence then follows from the dominated convergence theorem.

Taking $U = X_1$ here, by the self-similarity of the DLT we have that

$$D^\alpha(\mathcal{U}_m^0) \stackrel{\mathcal{D}}{=} U^\alpha D_{\{1\}}^\alpha(\mathcal{U}_N^0) + (1-U)^\alpha D_{\{2\}}^\alpha(\mathcal{U}_{m-1-N}^0) + U^\alpha, \quad (39)$$

where $N \sim \text{Bin}(m-1, U)$, given U , and, given U and N , $D_{\{1\}}^\alpha(\mathcal{U}_N^0)$ and $D_{\{2\}}^\alpha(\mathcal{U}_{m-1-N}^0)$ are independent with the distribution of $D^\alpha(\mathcal{U}_N^0)$ and $D^\alpha(\mathcal{U}_{m-1-N}^0)$, respectively. As $m \rightarrow \infty$, N and $m-N$ both tend to infinity almost surely, and so, by taking $m \rightarrow \infty$ in (39), we obtain the fixed-point equation (15).

The identity $E[D_\alpha] = (\alpha-1)^{-1}$ is obtained either from (24) of Proposition 3.1, or by taking expectations in (15). Next, if we set $\tilde{D}_\alpha = D_\alpha - E[D_\alpha]$, (15) yields (6). Then, using the definition (37) of $J(\alpha)$, the fact that $E[\tilde{D}_\alpha] = 0$, and independence, we obtain from (6) that

$$E[\tilde{D}_\alpha^2] = \frac{2E[\tilde{D}_\alpha^2]}{2\alpha+1} + \frac{\alpha^2+1}{(\alpha-1)^2(2\alpha+1)} + \frac{2\alpha J(\alpha)}{(\alpha-1)^2} - \frac{1}{(\alpha-1)^2},$$

and rearranging this gives (38). \square

Recall from Lemma 3.2 that \mathcal{L}_0^α is the limiting weight of edges attached to the origin in the DLT on uniform points. Combining this fact with Proposition 3.3, we obtain a similar result to the latter for the unrooted case as follows:

Proposition 3.4 *Let $\alpha > 1$. There is a random variable F_α , satisfying the distributional fixed-point equality (16), such that $D^\alpha(\mathcal{U}_m) \rightarrow F_\alpha$, as $n \rightarrow \infty$, almost surely and in L^2 . Further, $E[F_\alpha] = 1/(\alpha(\alpha - 1))$, and*

$$\text{Var}[F_\alpha] = \frac{1}{2\alpha} \text{Var}[D_\alpha] + \frac{\alpha + 2(2\alpha + 1)J(\alpha) - 2}{2\alpha^2(\alpha - 1)^2}, \quad (40)$$

where $J(\alpha)$ is given by (37) and $\text{Var}[D_\alpha]$ by (38).

Proof. By Lemma 3.2 and Proposition 3.3, there are random variables D_α and \mathcal{L}_0^α such that as $m \rightarrow \infty$ we have $D^\alpha(\mathcal{U}_m^0) \xrightarrow{L^2} D_\alpha$ and $\mathcal{L}_0^\alpha(\mathcal{U}_m^0) \xrightarrow{L^2} \mathcal{L}_0^\alpha$, also with almost sure convergence in both cases. Hence, setting $F_\alpha := D_\alpha - \mathcal{L}_0^\alpha$, we have by (27) that

$$D^\alpha(\mathcal{U}_m) = D^\alpha(\mathcal{U}_m^0) - \mathcal{L}_0^\alpha(\mathcal{U}_m^0) \rightarrow F_\alpha, \quad \text{a.s. and in } L^2. \quad (41)$$

Next, we show that F_α satisfies the distributional fixed-point equality (16). The self-similarity of the DLT implies that

$$D^\alpha(\mathcal{U}_m) \stackrel{\mathcal{D}}{=} U^\alpha D^\alpha(\mathcal{U}_N) + (1 - U)^\alpha D^\alpha(\mathcal{U}_{m-1-N}^0), \quad (42)$$

where $N \sim \text{Bin}(m-1, U)$, given U , and $D^\alpha(\mathcal{U}_N)$ and $D^\alpha(\mathcal{U}_{m-1-N}^0)$ are independent, given U and N . As $m \rightarrow \infty$, N and $m - N$ both tend to infinity almost surely, so taking $m \rightarrow \infty$ in (42), using Proposition 3.3 and eqn (41), we obtain the fixed-point equation (16).

The identity $E[F_\alpha] = \alpha^{-1}(\alpha - 1)^{-1}$ is obtained either by (33), or by taking expectations in (16) and using the formula for $E[D_\alpha]$ in Proposition 3.3. Then with $\tilde{F}_\alpha := F_\alpha - E[F_\alpha]$, we obtain (7) from (16), and using independence and the fact that $E[\tilde{F}_\alpha] = E[\tilde{D}_\alpha] = 0$ we obtain

$$\frac{2\alpha}{2\alpha + 1} E[\tilde{F}_\alpha^2] = \frac{E[\tilde{D}_\alpha^2]}{2\alpha + 1} + \frac{2\alpha J(\alpha) - 1}{\alpha^2(\alpha - 1)^2} + \frac{\alpha^2 + 1}{\alpha^2(\alpha - 1)^2(2\alpha + 1)},$$

which yields (40). \square

Examples. When $\alpha = 2$ we have that $E[D_2] = 1$ and $J(2) = 1/30$, so that $\text{Var}[D_2] = 2/9$. Also, $E[F_2] = 1/2$ and $\text{Var}[F_2] = 7/72 \approx 0.0972$.

3.4 Limit behaviour for $\alpha = 1$

Unlike in the case $\alpha > 1$, for $\alpha = 1$ the mean of the total weight $D^1(\mathcal{U}_m^0)$ diverges as $m \rightarrow \infty$ (see Proposition 3.1), so clearly there is no limiting distribution for $D^1(\mathcal{U}_m^0)$. Nevertheless, by using the orthogonality of the increments of the sequence $(D^1(\mathcal{U}_m^0), m \geq 1)$, we are able to show that the *centred* total weight $\tilde{D}^1(\mathcal{U}_m^0)$ does converge in distribution (in fact, in L^2) to a limiting random variable, and likewise for the unrooted case; this is our next result.

Subsequently, we shall characterize the distribution of the limiting random variable (for both the rooted and unrooted cases) by a fixed-point identity, and thereby complete the proof of Theorem 3.1 (i).

Proposition 3.5 (i) *As $m \rightarrow \infty$, the random variable $\tilde{D}^1(\mathcal{U}_m^0)$ converges in L^2 to a limiting random variable \tilde{D}_1 , with $E[\tilde{D}_1] = 0$ and $\text{Var}[\tilde{D}_1] = 2 - \pi^2/6$. In particular, $\text{Var}[D^1(\mathcal{U}_m^0)] \rightarrow 2 - \pi^2/6$ as $m \rightarrow \infty$.*

(ii) *As $m \rightarrow \infty$, $\tilde{D}^1(\mathcal{U}_m)$ converges in L^2 to the limiting random variable $\tilde{F}_1 := \tilde{D}_1 - \mathcal{L}_0^1 + 1$.*

Proof. Adopt the convention $D^1(\mathcal{U}_0^0) = 0$. By the orthogonality of the Z_j (Lemma 3.5) and (19), for $0 \leq \ell < m$,

$$\begin{aligned} \text{Var}[\tilde{D}^1(\mathcal{U}_m^0) - \tilde{D}^1(\mathcal{U}_\ell^0)] &= \text{Var} \sum_{j=\ell+1}^m (Z_j - E[Z_j]) \\ &= \sum_{j=\ell+1}^m \frac{j}{(j+1)^2(j+2)} \rightarrow 0 \text{ as } m, \ell \rightarrow \infty. \end{aligned}$$

Hence $\tilde{D}_1(\mathcal{U}_m^0)$ is a Cauchy sequence in L^2 , and so converges in L^2 to a limiting random variable, which we denote \tilde{D}_1 . Then $E[\tilde{D}_1] = \lim_{m \rightarrow \infty} E[\tilde{D}_1(\mathcal{U}_m^0)] = 0$, and

$$\begin{aligned} \text{Var}[\tilde{D}_1] &= \lim_{m \rightarrow \infty} \text{Var}[\tilde{D}^1(\mathcal{U}_m^0)] = \sum_{j=1}^{\infty} \frac{j}{(j+1)^2(j+2)} \\ &= \sum_{j=1}^{\infty} \left[\frac{2}{j+1} - \frac{2}{j+2} \right] - \sum_{j=1}^{\infty} \frac{1}{(j+1)^2} = 1 - \left(\frac{\pi^2}{6} - 1 \right) = 2 - \frac{\pi^2}{6}. \end{aligned}$$

It remains to prove part (ii), the convergence for the centred total length of the DLF $\tilde{D}^1(\mathcal{U}_m)$. We have by (27) that

$$\tilde{D}^1(\mathcal{U}_m) = \tilde{D}^1(\mathcal{U}_m^0) - \mathcal{L}_0^1(\mathcal{U}_m^0) + E[\mathcal{L}_0^1(\mathcal{U}_m^0)] \xrightarrow{L^2} \tilde{D}_1 - \mathcal{L}_0^1 + 1,$$

where the convergence follows by Lemma 3.2 and part (i). Thus $\tilde{D}^1(\mathcal{U}_m)$ converges in L^2 as $m \rightarrow \infty$. \square

For the next few results it is more convenient to consider the DLF defined on a Poisson number of points. Let (X_1, X_2, \dots) be a sequence of independent uniformly distributed random variables in $(0, 1]$, and let $(N(t), t \geq 0)$ be the counting

process of a homogeneous Poisson process of unit rate in $(0, \infty)$, independent of (X_1, X_2, \dots) . Thus $N(t)$ is a Poisson variable with parameter t . As before, let $\mathcal{U}_m = (X_1, \dots, X_m)$, and (for this section only) let $\mathcal{P}_t := \mathcal{U}_{N(t)}$. Let $\mathcal{P}_t^0 := \mathcal{U}_{N(t)}^0$, so that $\mathcal{P}_t^0 = (0, X_1, X_2, \dots, X_{N(t)})$.

We construct the DLF and DLT on $X_1, X_2, \dots, X_{N(t)}$ as before. Let $\tilde{D}^1(\mathcal{P}_t^0) = D^1(\mathcal{P}_t^0) - E[D^1(\mathcal{P}_t^0)]$ and $\tilde{D}^1(\mathcal{P}_t) = D^1(\mathcal{P}_t) - E[D^1(\mathcal{P}_t)]$. We aim to show that the limit distribution for $\tilde{D}^1(\mathcal{P}_t^0)$ is the same as for $\tilde{D}^1(\mathcal{U}_m^0)$, and likewise in the unrooted case. We shall need the following result.

Lemma 3.6 *As $t \rightarrow \infty$,*

$$\frac{d}{dt}E[D^1(\mathcal{P}_t)] = \frac{1}{t} + O(t^{-2}); \quad \text{and} \quad \frac{d}{dt}E[D^1(\mathcal{P}_t^0)] = \frac{1}{t} + O(t^{-2}). \quad (43)$$

Proof. The point set $\{X_1, \dots, X_{N(t)}\}$ is a homogeneous Poisson point process in $(0, 1)$, so we have

$$\begin{aligned} \frac{d}{dt}E[D^1(\mathcal{P}_t)] &= E[\text{length of new arrival}] \\ &= \int_0^1 du E[\text{dist. to next pt. to the left of } u \text{ in } \mathcal{P}_t] \\ &= \int_0^1 du \int_0^u s t e^{-ts} ds = \frac{1}{t} + \frac{2}{t^2} (e^{-t} - 1) + \frac{e^{-t}}{t} \\ &= \frac{1}{t} + O(t^{-2}). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d}{dt}E[D^1(\mathcal{P}_t^0)] &= \int_0^1 du E[\text{dist. to next pt. to the left of } u \text{ in } \mathcal{P}_t \cup \{0\}] \\ &= \int_0^1 du \int_0^u P[\text{dist. to next pt. to the left} > s] ds \\ &= \int_0^1 du \int_0^u e^{-ts} ds = \frac{1}{t} + \frac{e^{-t} - 1}{t^2} \\ &= \frac{1}{t} + O(t^{-2}). \quad \square \end{aligned}$$

Lemma 3.7 (i) *As $t \rightarrow \infty$, $\tilde{D}^1(\mathcal{P}_t^0)$ converges in distribution to \tilde{D}_1 , the L^2 large- m limit of $\tilde{D}^1(\mathcal{U}_m^0)$.*

(ii) *As $t \rightarrow \infty$, $\tilde{D}^1(\mathcal{P}_t)$ converges in distribution to \tilde{F}_1 , the L^2 large- m limit of $\tilde{D}^1(\mathcal{U}_m)$.*

Proof. (i) From Proposition 3.5, we have $\tilde{D}^1(\mathcal{U}_m^0) \xrightarrow{L^2} \tilde{D}_1$ as $m \rightarrow \infty$. Let $a_t := E[D^1(\mathcal{P}_t^0)]$ and $\mu_m := E[D^1(\mathcal{U}_m^0)]$. Since $\mu_m = E \sum_{i=1}^m Z_i = \sum_{i=1}^m (1+i)^{-1}$ by (19), for any positive integers ℓ, m we have

$$|\mu_m - \mu_\ell| = \sum_{j=\min(m, \ell)+1}^{\max(m, \ell)} \frac{1}{j+1} \leq \log \left(\frac{\max(m, \ell) + 1}{\min(m, \ell) + 1} \right) = \left| \log \left(\frac{m+1}{\ell+1} \right) \right|. \quad (44)$$

Note the distributional equalities

$$\begin{aligned}\mathcal{L}(D^1(\mathcal{P}_t^0)|N(t) = m) &= \mathcal{L}(D^1(\mathcal{U}_m^0)); \\ \mathcal{L}(D^1(\mathcal{P}_t^0) - \mu_{N(t)}|N(t) = m) &= \mathcal{L}(\tilde{D}^1(\mathcal{U}_m^0)).\end{aligned}\quad (45)$$

First we aim to show that $a_t - \mu_{\lfloor t \rfloor} \rightarrow 0$ as $t \rightarrow \infty$. Set $p_m(t) := e^{-t} \frac{t^m}{m!}$. Then we can write

$$\begin{aligned}a_t - \mu_{\lfloor t \rfloor} &= \sum_{m=0}^{\infty} p_m(t)(\mu_m - \mu_{\lfloor t \rfloor}) \\ &= \sum_{|m - \lfloor t \rfloor| \leq t^{3/4}} p_m(t)(\mu_m - \mu_{\lfloor t \rfloor}) + \sum_{|m - \lfloor t \rfloor| > t^{3/4}} p_m(t)(\mu_m - \mu_{\lfloor t \rfloor}).\end{aligned}\quad (46)$$

We examine these two sums separately. First consider the sum for $|m - \lfloor t \rfloor| \leq t^{3/4}$. By (44), we have

$$\begin{aligned}\sup_{m: |m - \lfloor t \rfloor| \leq t^{3/4}} |\mu_m - \mu_{\lfloor t \rfloor}| &\leq \max \left(\log \left(\frac{\lfloor t \rfloor + 1 + t^{3/4}}{\lfloor t \rfloor + 1} \right), \log \left(\frac{\lfloor t \rfloor + 1}{\lfloor t \rfloor + 1 - t^{3/4}} \right) \right) \\ &= O(t^{-1/4}) \rightarrow 0 \text{ as } t \rightarrow \infty.\end{aligned}$$

Hence the first sum in (46) tends to zero as $t \rightarrow \infty$. To estimate the second sum, observe that

$$\begin{aligned}\sum_{|m - \lfloor t \rfloor| > t^{3/4}} p_m(t)(\mu_m - \mu_{\lfloor t \rfloor}) &\leq \sum_{|m - \lfloor t \rfloor| > t^{3/4}} p_m(t)(m + t) \\ &= E \left[(N(t) + t) \mathbf{1}\{|N(t) - \lfloor t \rfloor| > t^{3/4}\} \right] \\ &\leq \left(E[(N(t) + t)^2] \cdot P[|N(t) - \lfloor t \rfloor| > t^{3/4}] \right)^{1/2}.\end{aligned}\quad (47)$$

By Chernoff bounds on the tail probabilities of a Poisson random variable (e.g. Lemma 1.4 of [14]), the expression (47) is $O(t \exp(-t^2/18))$ and so tends to zero. Hence the second sum in (46) tends to zero, and thus

$$a_t - \mu_{\lfloor t \rfloor} \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (48)$$

Now we show that $\tilde{D}^1(\mathcal{P}_t^0) \xrightarrow{\mathcal{D}} \tilde{D}_1$ as $t \rightarrow \infty$. We have

$$\tilde{D}^1(\mathcal{P}_t^0) = (D^1(\mathcal{P}_t^0) - \mu_{N(t)}) + (\mu_{N(t)} - \mu_{\lfloor t \rfloor}) + (\mu_{\lfloor t \rfloor} - a_t). \quad (49)$$

The final bracket tends to zero, by (48). Also, by (45) and the fact that $N(t) \rightarrow \infty$ a.s. as $t \rightarrow \infty$, we have

$$D^1(\mathcal{P}_t^0) - \mu_{N(t)} \xrightarrow{\mathcal{D}} \tilde{D}_1.$$

Finally, using (44), we have

$$|\mu_{N(t)} - \mu_{[t]}| \leq \left| \log \frac{N(t) + 1}{[t] + 1} \right| \xrightarrow{P} 0,$$

as $t \rightarrow \infty$, since $N(t)/[t] \xrightarrow{P} 1$. So Slutsky's theorem applied to (49) yields $\tilde{D}^1(\mathcal{P}_t^0) \xrightarrow{\mathcal{D}} \tilde{D}_1$ as $t \rightarrow \infty$, completing the proof of (i)

The proof of (ii) follows in the same way as that of (i), except that in (44) the first equals sign is replaced by an inequality \leq . This does not affect the rest of the proof. \square

The next two propositions complete the proof of Theorem 3.1.

Proposition 3.6 *The limiting random variable \tilde{D}_1 of Proposition 3.5 (i) satisfies the fixed-point equation (5).*

Proof. For integer $n > 0$, let $T_n := \min\{s : N(s) \geq n\}$, the n th arrival time of the Poisson process with counting process $N(\cdot)$. Set $T := T_1$, and set $U := X_1$ (which is uniform on $(0, 1)$).

By the Marking Theorem for Poisson processes [10], the two-dimensional point process $\mathcal{Q} := \{(X_n, T_n) : n \geq 1\}$ is a homogeneous Poisson process of unit intensity on $(0, 1) \times (0, \infty)$. Given the value of (U, T) , the restriction of \mathcal{Q} to $(0, U] \times (T, \infty)$ and the restriction of \mathcal{Q} to $(U, 1] \times (T, \infty)$ are independent homogeneous Poisson processes on these regions. Hence, by scaling properties of the Poisson process (see the Mapping Theorem in [10]) and of the DLT, writing $D_{\{i\}}^1(\cdot)$, $i = 1, 2$ for independent copies of $D^1(\cdot)$, we have

$$D^1(\mathcal{P}_t^0) \stackrel{\mathcal{D}}{=} \left(U D_{\{1\}}^1(\mathcal{P}_{U(t-T)}^0) + (1 - U) D_{\{2\}}^1(\mathcal{P}_{(1-U)(t-T)}^0) + U \right) \mathbf{1}\{t > T\}. \quad (50)$$

Let $a_s = 0$ for $s \leq 0$, and $a_s = E[D^1(\mathcal{P}_s^0)]$ for $s > 0$. Then $\tilde{D}^1(\mathcal{P}_t^0) = D^1(\mathcal{P}_t^0) - a_t$, so that by (50),

$$\begin{aligned} \tilde{D}^1(\mathcal{P}_t^0) &\stackrel{\mathcal{D}}{=} \left(U \tilde{D}_{\{1\}}^1(\mathcal{P}_{U(t-T)}^0) + (1 - U) \tilde{D}_{\{2\}}^1(\mathcal{P}_{(1-U)(t-T)}^0) + U \right) \mathbf{1}\{t > T\} \\ &\quad + U (a_{U(t-T)} - a_t) + (1 - U) (a_{(1-U)(t-T)} - a_t). \end{aligned} \quad (51)$$

From Lemma 3.6 we have $\frac{da_t}{dt} = \frac{1}{t} + O(t^{-2})$. Hence, if $T < t$, then

$$a_t - a_{U(t-T)} = \int_{U(t-T)}^t \frac{da_s}{ds} ds = \log t - \log\{U(t-T)\} + O((U(t-T))^{-1}),$$

and hence as $t \rightarrow \infty$,

$$a_t - a_{U(t-T)} \rightarrow -\log U, \quad \text{a.s..} \quad (52)$$

Since $P[T < t]$ tends to 1, by making $t \rightarrow \infty$ in (51) and using Slutsky's theorem we obtain (5). \square

Proposition 3.7 *The limiting random variable \tilde{F}_1 of Proposition 3.5 (ii) satisfies the fixed-point equation (5), and so has the same distribution as \tilde{D}_1 . Also, $\text{Cov}(\tilde{F}_1, \tilde{D}_1) = (7/4) - \pi^2/6$.*

Proof. The proof follows similar lines to that of Proposition 3.6. Once more let $a_s = E[D^1(\mathcal{P}_s^0)]$, for $s \geq 0$, and $a_s = 0$ for $s < 0$. Let $b_s = E[D^1(\mathcal{P}_s)]$ for $s > 0$, and $b_s = 0$ for $s \leq 0$, and let $T := \min\{t : N(t) \geq 1\}$, Then

$$D^1(\mathcal{P}_t) \stackrel{\mathcal{D}}{=} \left(U D_{\{1\}}^1(\mathcal{P}_{U(t-T)}) + (1-U) D_{\{2\}}^1(\mathcal{P}_{(1-U)(t-T)}^0) \right) \mathbf{1}\{t > T\}, \quad (53)$$

where $D_{\{1\}}^1(\cdot)$ and $D_{\{2\}}^1(\cdot)$ are independent copies of $D^1(\cdot)$. Then $\tilde{D}^1(\mathcal{P}_t) = D^1(\mathcal{P}_t) - b_t$ and $\tilde{D}^1(\mathcal{P}_t^0) = D^1(\mathcal{P}_t^0) - a_t$, so that (53) yields

$$\begin{aligned} \tilde{D}^1(\mathcal{P}_t) &\stackrel{\mathcal{D}}{=} \left(U \tilde{D}_{\{1\}}^1(\mathcal{P}_{U(t-T)}) + (1-U) \tilde{D}_{\{2\}}^1(\mathcal{P}_{(1-U)(t-T)}^0) \right) \mathbf{1}\{t > T\} \\ &\quad + U (b_{U(t-T)} - b_t) + (1-U) (a_{(1-U)(t-T)} - b_t). \end{aligned} \quad (54)$$

From Lemma 3.6 we have $\frac{db_t}{dt} = \frac{1}{t} + O(t^{-2})$. Hence, by the same argument as used at (52),

$$b_t - b_{U(t-T)} \rightarrow -\log U \quad \text{a.s.}$$

Also, $a_t - b_t = E[\mathcal{L}_0^1(\mathcal{P}_t^0)]$ by (27), so that $\lim_{t \rightarrow \infty} (a_t - b_t) = 1$, by Lemma 3.2 and the fact that $E[\mathcal{L}_0^1] = 1$ (eqn (29)). Using also (52) we find that as $t \rightarrow \infty$,

$$a_{(1-U)(t-T)} - b_t = (a_{(1-U)(t-T)} - a_t) + (a_t - b_t) \rightarrow 1 + \log(1-U), \quad \text{a.s.}$$

Taking $t \rightarrow \infty$ in (54), and using Slutsky's theorem, we obtain

$$\tilde{F}_1 \stackrel{\mathcal{D}}{=} U \tilde{F}_1 + (1-U) \tilde{D}_1 + U \log U + (1-U) \log(1-U) + (1-U). \quad (55)$$

The change of variable $(1-U) \mapsto U$ then shows that \tilde{D}_1 as defined at (5) satisfies (55), and so by the uniqueness of solution, \tilde{F}_1 has the same distribution as \tilde{D}_1 and satisfies (5).

To obtain the covariance of \tilde{F}_1 and \tilde{D}_1 , observe from Proposition 3.5 (ii) that $\mathcal{L}_0^1 = \tilde{D}_1 - \tilde{F}_1 + 1$, and therefore by (29), we have that

$$1/2 = \text{Var}[\mathcal{L}_0^1] = \text{Var}[\tilde{D}_1] + \text{Var}[\tilde{F}_1] - 2\text{Cov}(\tilde{D}_1, \tilde{F}_1). \quad (56)$$

Since $\text{Var}[\tilde{F}_1] = \text{Var}[\tilde{D}_1] = 2 - \pi^2/6$ by Proposition 3.5 (i), rearranging (56) we find that $\text{Cov}(\tilde{D}_1, \tilde{F}_1) = (7/4) - \pi^2/6$. \square

Remark. Figure 2 is a plot of the estimated probability density function of \tilde{D}_1 . This was obtained by performing 10^6 repeated simulations of the DLT on a sequence of 10^3 uniform (simulated) random points on $(0, 1]$. For each simulation, the expected value of $D^1(\mathcal{U}_{10^3})$ (which is precisely $(1/2) + (1/3) + \dots + (1/1001)$ by Lemma 3.1) was subtracted from the total length of the simulated DLT to give an approximate realization of \tilde{D}_1 . The density function was then estimated from the sample of 10^6 approximate realizations of \tilde{D}_1 , using a window width of 0.0025. The simulated sample from which the density estimate for \tilde{D}_1 was taken had sample mean $\approx -2 \times 10^{-4}$ and sample variance ≈ 0.3543 , which are reasonably close to the expectation and variance of \tilde{D}_1 .

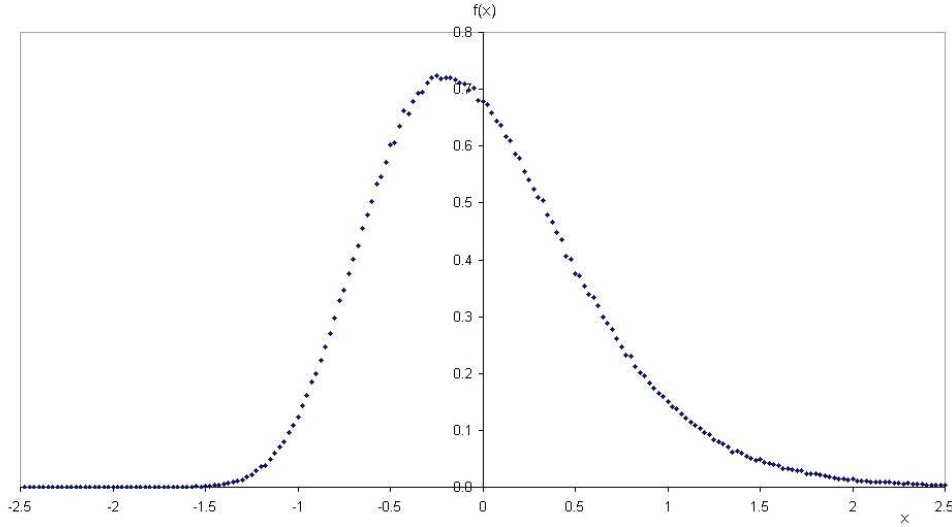


Figure 2: Estimated probability density function for \tilde{D}_1 .

4 General results in geometric probability

Notions of *stabilizing* functionals of point sets have recently proved to be a useful basis for a general methodology for establishing limit theorems for functionals of random point sets in \mathbf{R}^d . In particular, Penrose and Yukich [17, 18] provide general central limit theorems and laws of large numbers for stabilizing functionals. One might hope to apply these results in the case of the MDSF weight. In fact we shall obtain our law of large numbers (Theorem 2.1) by application of a result from [18], but to obtain the central limit theorem for edges away from the boundary in the MDSF and MDST, we need an extension of the general result in [17]. It is these general results that we describe in the present section.

For our general results, we use the following notation. Let $d \geq 1$ be an integer. For $\mathcal{X} \subset \mathbf{R}^d$, constant $a > 0$, and $\mathbf{y} \in \mathbf{R}^d$, let $\mathbf{y} + a\mathcal{X}$ denote the transformed set $\{\mathbf{y} + a\mathbf{x} : \mathbf{x} \in \mathcal{X}\}$. Let $\text{diam}(\mathcal{X}) := \sup\{\|\mathbf{x}_1 - \mathbf{x}_2\| : \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}\}$, and let $\text{card}(\mathcal{X})$ denote the cardinality (number of elements) of \mathcal{X} (when finite).

For $\mathbf{x} \in \mathbf{R}^d$ and $r > 0$, let $B(\mathbf{x}; r)$ denote the closed Euclidean ball with centre \mathbf{x} and radius r , and let $Q(\mathbf{x}; r)$ denote the corresponding l_∞ ball, i.e., the d -cube $\mathbf{x} + [-r, r]^d$. For bounded measurable $R \subset \mathbf{R}^d$ let $|R|$ denote the Lebesgue measure of R , let ∂R denote the topological boundary of R and for $r > 0$, set $\partial_r R := \cup_{\mathbf{x} \in \partial R} Q(\mathbf{x}; r)$, the r -neighbourhood of the boundary of R .

4.1 A general law of large numbers

Let $\xi(\mathbf{x}; \mathcal{X})$ be a measurable \mathbf{R}_+ -valued function defined for all pairs $(\mathbf{x}, \mathcal{X})$, where $\mathcal{X} \subset \mathbf{R}^d$ is finite and $\mathbf{x} \in \mathcal{X}$. Assume ξ is translation invariant, that is, for all $\mathbf{y} \in \mathbf{R}^d$, $\xi(\mathbf{y} + \mathbf{x}; \mathbf{y} + \mathcal{X}) = \xi(\mathbf{x}; \mathcal{X})$. When $\mathbf{x} \notin \mathcal{X}$, we abbreviate the notation

$\xi(\mathbf{x}; \mathcal{X} \cup \{\mathbf{x}\})$ to $\xi(\mathbf{x}; \mathcal{X})$.

For our general law of large numbers, we use a notion of stabilization defined as follows. For any locally finite point set $\mathcal{X} \subset \mathbf{R}^d$ and any $\ell \in \mathbf{N}$ define

$$\xi^+(\mathcal{X}; \ell) := \sup_{k \in \mathbf{N}} \left(\operatorname{ess\,sup}_{\ell, k} \{ \xi(\mathbf{0}; (\mathcal{X} \cap B(\mathbf{0}; \ell)) \cup \mathcal{A} \} \right), \text{ and}$$

$$\xi^-(\mathcal{X}; \ell) := \inf_{k \in \mathbf{N}} \left(\operatorname{ess\,inf}_{\ell, k} \{ \xi(\mathbf{0}; (\mathcal{X} \cap B(\mathbf{0}; \ell)) \cup \mathcal{A} \} \right);$$

where $\operatorname{ess\,sup}_{\ell, k}$ is the essential supremum, with respect to Lebesgue measure on \mathbf{R}^{dk} , over sets $\mathcal{A} \subset \mathbf{R}^d \setminus B(\mathbf{0}; \ell)$ of cardinality k . Define the *limit* of ξ on \mathcal{X} by

$$\xi_\infty(\mathcal{X}) := \limsup_{k \rightarrow \infty} \xi^+(\mathcal{X}; k).$$

We say the functional ξ *stabilizes* on \mathcal{X} if

$$\lim_{k \rightarrow \infty} \xi^+(\mathcal{X}; k) = \lim_{k \rightarrow \infty} \xi^-(\mathcal{X}; k) = \xi_\infty(\mathcal{X}). \quad (57)$$

For $\tau \in (0, \infty)$, let \mathcal{H}_τ be a homogeneous Poisson process of intensity τ on \mathbf{R}^d . The following general law of large numbers is due to Penrose and Yukich [18]. We shall use it to prove Theorem 2.1.

Lemma 4.1 [18] *Suppose $q = 1$ or $q = 2$. Suppose ξ is almost surely stabilizing on \mathcal{H}_τ , with limit $\xi_\infty(\mathcal{H}_\tau)$, for all $\tau \in (0, \infty)$. Let f be a probability density function on \mathbf{R}^d , and let \mathcal{X}_n be the point process consisting of n independent random d -vectors with common density f . If ξ satisfies the moments condition*

$$\sup_{n \in \mathbf{N}} E \left[\xi \left(n^{1/d} \mathbf{X}_1; n^{1/d} \mathcal{X}_n \right)^p \right] < \infty, \quad (58)$$

for some $p > q$, then as $n \rightarrow \infty$,

$$n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi(n^{1/d} \mathbf{x}; n^{1/d} \mathcal{X}_n) \xrightarrow{L^q} \int_{\mathbf{R}^d} E [\xi_\infty(\mathcal{H}_{f(\mathbf{x})})] f(\mathbf{x}) d\mathbf{x}, \quad (59)$$

and the limit is finite.

4.2 General central limit theorems

In the course of the proof of Theorem 2.2, we shall use a modified form of a general central limit theorem obtained for functionals of geometric graphs by Penrose and Yukich [17]. We recall the setup of [17]. As in Section 4.1, let $\xi(\mathbf{x}; \mathcal{X})$ be a translation invariant real-valued functional defined for finite $\mathcal{X} \subset \mathbf{R}^d$ and $\mathbf{x} \in \mathcal{X}$. Then ξ induces a translation invariant functional $H(\mathcal{X}; S)$ defined on all finite point sets $\mathcal{X} \subset \mathbf{R}^d$ and all Borel-measurable regions $S \subseteq \mathbf{R}^d$ by

$$H(\mathcal{X}; S) := \sum_{\mathbf{x} \in \mathcal{X} \cap S} \xi(\mathbf{x}; \mathcal{X}). \quad (60)$$

It is this ‘restricted’ functional that interests us here, while [17] is concerned rather with the global functional $H(\mathcal{X}; \mathbf{R}^d)$. In our particular application (the length of edges of the MDST on random points in a square), the global functional fails to satisfy the conditions of the central limit theorems in [17], owing to boundary effects. Here we generalize the result in [17] to the ‘restricted’ functional $H(\mathcal{X}; S)$. It is this generalized result that we can apply to the MDST, when we take S to be a region ‘away from the boundary’ of the square in which the random points are placed.

We use a notion of stabilization for H which is related to, but not equivalent to, the notion of stabilization of ξ used in Section 4.1. Loosely speaking, ξ is stabilizing if when a point is inserted at the origin into a homogeneous Poisson process, only nearby Poisson points affect the inserted point; for H to be stabilizing we require also that the inserted point affects only nearby points.

For $B \subseteq \mathbf{R}^d$, let $\Delta(\mathcal{X}; B)$ denote the ‘add one cost’ of the functional H on the insertion of a point at the origin,

$$\Delta(\mathcal{X}; B) := H(\mathcal{X} \cup \{\mathbf{0}\}; B) - H(\mathcal{X}; B).$$

Let $\mathcal{P} := \mathcal{H}_1$ (a homogeneous Poisson point process of unit intensity on \mathbf{R}^d). Let $\mathcal{Q}_n := \mathcal{P} \cap R_n$ (the restriction of \mathcal{P} to R_n). Adapting the ideas of [17], we make the following definitions.

Definition 4.1 *We say the functional H is strongly stabilizing if there exist almost surely finite random variables R (a radius of stabilization) and $\Delta(\infty)$ such that, with probability 1, for any $B \supseteq B(\mathbf{0}; R)$,*

$$\Delta(\mathcal{P} \cap B(\mathbf{0}; R) \cup \mathcal{A}; B) = \Delta(\infty), \quad \forall \text{ finite } \mathcal{A} \subset \mathbf{R}^d \setminus B(\mathbf{0}; R).$$

We say that the functional H is *polynomially bounded* if, for all $B \ni \mathbf{0}$, there exists a constant β such that for all finite sets $\mathcal{X} \subset \mathbf{R}^d$,

$$|H(\mathcal{X}; B)| \leq \beta (\text{diam}(\mathcal{X}) + \text{card}(\mathcal{X}))^\beta. \quad (61)$$

We say that H is *homogeneous of order γ* if for all finite $\mathcal{X} \subset \mathbf{R}^d$ and Borel $B \subseteq \mathbf{R}^d$, and all $a \in \mathbf{R}$, $H(a\mathcal{X}; aB) = a^\gamma H(\mathcal{X}; B)$.

Let (R_n, S_n) , for $n = 1, 2, \dots$, be a sequence of ordered pairs of bounded Borel subsets of \mathbf{R}^d , such that $S_n \subseteq R_n$ for all n . Assume that for all $r > 0$, $n^{-1}|\partial_r R_n| \rightarrow 0$ and $n^{-1}|\partial_r S_n| \rightarrow 0$ (the *vanishing relative boundary condition*). Assume also that $|R_n| = n$ for all n , and $|S_n|/n \rightarrow 1$ as $n \rightarrow \infty$; that S_n tends to \mathbf{R}^d , in the sense that $\bigcup_{n \geq 1} \bigcap_{m \geq n} S_m = \mathbf{R}^d$; and that there exists a constant β such that $\text{diam}(R_n) \leq \beta n^\beta$ for all n (the *polynomial boundedness condition* on $(R_n, S_n)_{n \geq 1}$). Subject to these conditions, the choice of $(R_n, S_n)_{n \geq 1}$ is arbitrary.

Let $\mathbf{U}_{1,n}, \mathbf{U}_{2,n}, \dots$ be i.i.d. uniform random vectors on R_n . Let

$$\mathcal{U}_{m,n} = \{\mathbf{U}_{1,n}, \dots, \mathbf{U}_{m,n}\}$$

(a binomial point process), and for Borel $A \subseteq \mathbf{R}^d$ with $0 < |A| < \infty$, let $\mathcal{U}_{m,A}$ be the binomial point process of m i.i.d. uniform random vectors on A .

Let \mathcal{R} be the collection of all pairs (A, B) with $A, B \subset \mathbf{R}^d$ of the form $(A, B) = (\mathbf{x} + R_n, \mathbf{x} + S_n)$ with $\mathbf{x} \in \mathbf{R}^d$ and $n \in \mathbf{N}$. That is, \mathcal{R} is the collection of all the (R_n, S_n) and their translates.

We say that the functional H satisfies the *uniform bounded moments condition* on \mathcal{R} if

$$\sup_{(A,B) \in \mathcal{R}: \mathbf{0} \in A} \left(\sup_{|A|/2 \leq m \leq 3|A|/2} \{E[\Delta(\mathcal{U}_{m,A}; B)^4]\} \right) < \infty. \quad (62)$$

We now state the general results, which extend those of Penrose and Yukich (Theorem 2.1 and Corollary 2.1 in [17]).

Theorem 4.1 *Suppose that H is strongly stabilizing, is polynomially bounded (61), and satisfies the uniform bounded moments condition (62) on \mathcal{R} . Then there exist constants s^2, t^2 , with $0 \leq t^2 \leq s^2$, such that as $n \rightarrow \infty$,*

- (i) $n^{-1} \text{Var}(H(\mathcal{Q}_n; S_n)) \rightarrow s^2$;
- (ii) $n^{-1/2} (H(\mathcal{Q}_n; S_n) - E[H(\mathcal{Q}_n; S_n)]) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2)$;
- (iii) $n^{-1} \text{Var}(H(\mathcal{U}_{n,n}; S_n)) \rightarrow t^2$;
- (iv) $n^{-1/2} (H(\mathcal{U}_{n,n}; S_n) - E[H(\mathcal{U}_{n,n}; S_n)]) \xrightarrow{\mathcal{D}} \mathcal{N}(0, t^2)$.

Also, s^2 and t^2 are independent of the choice of the (R_n, S_n) . Further, if the distribution of $\Delta(\infty)$ is nondegenerate, then $s^2 \geq t^2 > 0$.

Let R_0 be a fixed bounded Borel subset of \mathbf{R}^d with $|R_0| = 1$ and $|\partial R_0| = 0$. Let $(S_{0,n}, n \geq 1)$ be a sequence of Borel sets with $S_{0,n} \subseteq R_0$ such that $|S_{0,n}| \rightarrow 1$ as $n \rightarrow \infty$ and for all $r > 0$ we have $|\partial_{n^{-1/d}r} S_{0,n}| \rightarrow 0$ as $n \rightarrow \infty$.

Let \mathcal{R}_0 be the collection of all pairs of the form $(\mathbf{x} + n^{1/d}R_0, \mathbf{x} + n^{1/d}S_{0,n})$ with $n \geq 1$ and $\mathbf{x} \in \mathbf{R}^d$. Let \mathcal{X}_n be the binomial point process of n i.i.d. uniform random vectors on R_0 , and let \mathcal{P}_n be a homogeneous Poisson point process of intensity n on R_0 .

Corollary 4.1 *Suppose H is strongly stabilizing, satisfies the uniform bounded moments condition on \mathcal{R}_0 , is polynomially bounded and is homogeneous of order γ . Then with s^2, t^2 as in Theorem 4.1 we have that, as $n \rightarrow \infty$*

- (i) $n^{(2\gamma/d)-1} \text{Var}(H(\mathcal{P}_n; S_{0,n})) \rightarrow s^2$;
- (ii) $n^{(\gamma/d)-1/2} (H(\mathcal{P}_n; S_{0,n}) - E[H(\mathcal{P}_n; S_{0,n})]) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2)$;
- (iii) $n^{(2\gamma/d)-1} \text{Var}(H(\mathcal{X}_n; S_{0,n})) \rightarrow t^2$;
- (iv) $n^{(\gamma/d)-1/2} (H(\mathcal{X}_n; S_{0,n}) - E[H(\mathcal{X}_n; S_{0,n})]) \xrightarrow{\mathcal{D}} \mathcal{N}(0, t^2)$.

Proof. The corollary follows from Theorem 4.1 by taking $R_n = n^{1/d}R_0$ and $S_n = n^{1/d}S_{0,n}$ (or suitable translates thereof), and scaling, since H is homogeneous of order γ . \square

4.3 Proof of Theorem 4.1: the Poisson case

Let \mathcal{P} be a Poisson process of unit intensity on \mathbf{R}^d . We say the functional H is *weakly stabilizing* on \mathcal{R} if there is a random variable $\Delta(\infty)$ such that

$$\Delta(\mathcal{P} \cap A; B) \xrightarrow{\text{a.s.}} \Delta(\infty), \quad (63)$$

as $(A, B) \rightarrow \mathbf{R}^d$ through \mathcal{R} , by which we mean (63) holds whenever (A, B) is an \mathcal{R} -valued sequence of the form $(A_n, B_n)_{n \geq 1}$, such that $\cup_{n \geq 1} \cap_{m \geq n} B_m = \mathbf{R}^d$. Note that strong stabilization of H implies weak stabilization of H .

We say the functional H satisfies the *Poisson bounded moments condition* on \mathcal{R} if

$$\sup_{(A, B) \in \mathcal{R}: \mathbf{0} \in A} \{E[\Delta(\mathcal{P} \cap A; B)^4]\} < \infty. \quad (64)$$

Theorem 4.2 *Suppose that H is weakly stabilizing on \mathcal{R} (63) and satisfies (64). Then there exists $s^2 \geq 0$ such that as $n \rightarrow \infty$, $n^{-1} \text{Var}[H(\mathcal{Q}_n; S_n)] \rightarrow s^2$ and $n^{-1/2}(H(\mathcal{Q}_n; S_n) - E[H(\mathcal{Q}_n; S_n)]) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2)$.*

Before proving Theorem 4.2, we require further definitions and a lemma. Let \mathcal{P}' be an independent copy of the Poisson process \mathcal{P} . For $\mathbf{x} \in \mathbf{Z}^d$, set

$$\mathcal{P}''(\mathbf{x}) = (\mathcal{P} \setminus Q(\mathbf{x}; 1/2)) \cup (\mathcal{P}' \cap Q(\mathbf{x}; 1/2)).$$

Then given a translation invariant functional H on point sets in \mathbf{R}^d , define

$$\Delta_{\mathbf{x}}(A; B) := H(\mathcal{P}''(\mathbf{x}) \cap A; B) - H(\mathcal{P} \cap A; B);$$

this is the change in $H(\mathcal{P} \cap A; B)$ when the Poisson points in $Q(\mathbf{x}; 1/2)$ are resampled.

Lemma 4.2 *Suppose H is weakly stabilizing on \mathcal{R} . Then for all $\mathbf{x} \in \mathbf{Z}^d$, there is a random variable $\Delta_{\mathbf{x}}(\infty)$ such that for all $\mathbf{x} \in \mathbf{Z}^d$,*

$$\Delta_{\mathbf{x}}(A; B) \xrightarrow{\text{a.s.}} \Delta_{\mathbf{x}}(\infty), \quad (65)$$

as $(A, B) \rightarrow \mathbf{R}^d$ through \mathcal{R} . Moreover, if H satisfies (64), then

$$\sup_{(A, B) \in \mathcal{R}, \mathbf{x} \in \mathbf{Z}^d} E[(\Delta_{\mathbf{x}}(A; B))^4] < \infty. \quad (66)$$

Proof. Set $C_0 = Q(\mathbf{0}; 1/2)$. By translation invariance, we need only consider the case $\mathbf{x} = \mathbf{0}$, and thus it suffices to prove that the variables $H(\mathcal{P} \cap A; B) - H(\mathcal{P} \cap A \setminus C_0; B)$ converge almost surely as $(A, B) \rightarrow \mathbf{R}^d$ through \mathcal{R} .

The number N of points of \mathcal{P} in C_0 is Poisson with parameter 1. Let $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_N$ be the points of $\mathcal{P} \cap C_0$, taken in an order chosen uniformly at random from the $N!$ possibilities. Then, provided $C_0 \subseteq A$,

$$H(\mathcal{P} \cap A; B) - H(\mathcal{P} \cap A \setminus C_0; B) = \sum_{i=0}^{N-1} \delta_i(A; B),$$

where

$$\delta_i(A; B) := H((\mathcal{P} \cap A \setminus C_0) \cup \{\mathbf{V}_1, \dots, \mathbf{V}_{i+1}\}; B) - H((\mathcal{P} \cap A \setminus C_0) \cup \{\mathbf{V}_1, \dots, \mathbf{V}_i\}; B).$$

Since N is a.s. finite, it suffices to prove that each $\delta_i(A; B)$ converges almost surely as $(A, B) \rightarrow \mathbf{R}^d$ through \mathcal{R} . Let \mathbf{U} be a uniform random vector on C_0 , independent of \mathcal{P} . The distribution of the translated point process $-\mathbf{V}_{i+1} + \{\mathbf{V}_1, \dots, \mathbf{V}_i\} \cup (\mathcal{P} \setminus C_0)$ is the same as the conditional distribution of \mathcal{P} given that the number of points in $-\mathbf{U} + C_0$ is equal to i , an event of strictly positive probability. By assumption, this satisfies weak stabilization, which proves (65).

Next we prove (66). If $Q(\mathbf{x}; 1/2) \cap A = \emptyset$ then $\Delta_{\mathbf{x}}(A; B)$ is zero with probability 1. By translation invariance, it suffices to consider the $\mathbf{x} = \mathbf{0}$ case, that is, to prove

$$\sup_{(A, B) \in \mathcal{R}: C_0 \cap A \neq \emptyset} E \left[(\Delta_{\mathbf{0}}(A; B))^4 \right] < \infty. \quad (67)$$

The proof of this now follows the proof of (3.4) of [17], but with $\delta_i(A)$ replaced by $\delta_i(A; B)$ everywhere. \square

Proof of Theorem 4.2. Here we can assume, without loss of generality, that $\mathcal{Q}_n = \mathcal{P} \cap R_n$. For $\mathbf{x} \in \mathbf{Z}^d$, let $\mathcal{F}_{\mathbf{x}}$ denote the σ -field generated by the points of \mathcal{P} in $\cup_{\mathbf{y} \in \mathbf{Z}^d: \mathbf{y} \leq \mathbf{x}} Q(\mathbf{y}; 1/2)$, where the order in the union is the lexicographic order on \mathbf{Z}^d .

Let R'_n be the set of points $\mathbf{x} \in \mathbf{Z}^d$ such that $Q(\mathbf{x}; 1/2) \cap R_n \neq \emptyset$. Let $k_n = \text{card}(R'_n)$. Then we have that

$$R_n \subseteq \bigcup_{\mathbf{x} \in R'_n} Q(\mathbf{x}; 1/2) \subseteq R_n \cup \partial_1(R_n),$$

so that

$$|R_n| \leq k_n \leq |R_n| + |\partial_1(R_n)|.$$

The vanishing relative boundary condition then implies that $k_n/n \rightarrow 1$ as $n \rightarrow \infty$.

Define the filtration $(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_{k_n})$ as follows: let \mathcal{G}_0 be the trivial σ -field, label the elements of R'_n in lexicographic order as $\mathbf{x}_1, \dots, \mathbf{x}_{k_n}$ and let $\mathcal{G}_i = \mathcal{F}_{\mathbf{x}_i}$ for $1 \leq i \leq k_n$. Then $H(\mathcal{Q}_n; S_n) - E[H(\mathcal{Q}_n; S_n)] = \sum_{i=1}^{k_n} D_i$, where we set

$$D_i = E[H(\mathcal{Q}_n; S_n) | \mathcal{G}_i] - E[H(\mathcal{Q}_n; S_n) | \mathcal{G}_{i-1}] = E[-\Delta_{\mathbf{x}_i}(R_n; S_n) | \mathcal{F}_{\mathbf{x}_i}]. \quad (68)$$

By orthogonality of martingale differences, $\text{Var}[H(\mathcal{Q}_n; S_n)] = E \sum_{i=1}^{k_n} D_i^2$. By this fact, along with a CLT for martingale differences (Theorem 2.3 of [11] or Theorem 2.10 of [14]), it suffices to prove the conditions

$$\sup_{n \geq 1} E \left[\max_{1 \leq i \leq k_n} \left\{ k_n^{-1/2} |D_i| \right\}^2 \right] < \infty, \quad (69)$$

$$k_n^{-1/2} \max_{1 \leq i \leq k_n} |D_i| \xrightarrow{P} 0, \quad (70)$$

and for some $s^2 \geq 0$,

$$k_n^{-1} \sum_{i=1}^{k_n} D_i^2 \xrightarrow{L^1} s^2. \quad (71)$$

Using (66), and the representation (68) for D_i , we can verify (69) and (70) in just the same manner as for the equivalent estimates (3.7) and (3.8) in [17].

We now prove (71). By (65), for each $\mathbf{x} \in \mathbf{Z}^d$ the variables $\Delta_{\mathbf{x}}(A; B)$ converge almost surely to a limit, denoted $\Delta_{\mathbf{x}}(\infty)$, as $(A, B) \rightarrow \mathbf{R}^d$ through \mathcal{R} . For $\mathbf{x} \in \mathbf{Z}^d$ and $(A, B) \in \mathcal{R}$, let

$$F_{\mathbf{x}}(A; B) = E[\Delta_{\mathbf{x}}(A; B) | \mathcal{F}_{\mathbf{x}}]; \quad F_{\mathbf{x}} = E[\Delta_{\mathbf{x}}(\infty) | \mathcal{F}_{\mathbf{x}}].$$

Then $(F_{\mathbf{x}}, \mathbf{x} \in \mathbf{Z}^d)$ is a stationary family of random variables. Set $s^2 = E[F_{\mathbf{0}}^2]$. We claim that the ergodic theorem implies

$$k_n^{-1} \sum_{\mathbf{x} \in R'_n} F_{\mathbf{x}}^2 \xrightarrow{L^1} s^2. \quad (72)$$

The proof of this follows, with minor modifications, the proof of the corresponding result (3.10) in [17].

We need to show that $F_{\mathbf{x}}(R_n; S_n)^2$ approximates to $F_{\mathbf{x}}^2$. We consider \mathbf{x} at the origin $\mathbf{0}$. For any $(A, B) \in \mathcal{R}$, by Cauchy-Schwarz,

$$E[|F_{\mathbf{0}}(A; B)^2 - F_{\mathbf{0}}^2|] \leq (E[(F_{\mathbf{0}}(A; B) + F_{\mathbf{0}})^2])^{1/2} (E[(F_{\mathbf{0}}(A; B) - F_{\mathbf{0}})^2])^{1/2}. \quad (73)$$

By the definition of $F_{\mathbf{0}}$ and the conditional Jensen inequality,

$$\begin{aligned} E[(F_{\mathbf{0}}(A; B) + F_{\mathbf{0}})^2] &= E \left[(E[\Delta_{\mathbf{0}}(A; B) + \Delta_{\mathbf{0}}(\infty) | \mathcal{F}_{\mathbf{0}}])^2 \right] \\ &\leq E[E[(\Delta_{\mathbf{0}}(A; B) + \Delta_{\mathbf{0}}(\infty))^2 | \mathcal{F}_{\mathbf{0}}]] \\ &= E[(\Delta_{\mathbf{0}}(A; B) + \Delta_{\mathbf{0}}(\infty))^2], \end{aligned}$$

which is uniformly bounded by (65) and (66). Similarly,

$$E[(F_{\mathbf{0}}(A; B) - F_{\mathbf{0}})^2] \leq E[(\Delta_{\mathbf{0}}(A; B) - \Delta_{\mathbf{0}}(\infty))^2], \quad (74)$$

which is also uniformly bounded by (65) and (66). For any \mathcal{R} -valued sequence $(A_n, B_n)_{n \geq 1}$ with $\cup_{n \geq 1} \cap_{m \geq n} B_n = \mathbf{R}^d$, the sequence $(\Delta_{\mathbf{0}}(A_n; B_n) - \Delta_{\mathbf{0}}(\infty))^2$ tends to 0 almost surely by (65), and is uniformly integrable by (66), and therefore the expression (74) tends to zero so that by (73), $E[|F_{\mathbf{0}}(A_n; B_n)^2 - F_{\mathbf{0}}^2|] \rightarrow 0$.

Returning to the given sequence (R_n, S_n) , let $\varepsilon > 0$. By the vanishing relative boundary condition, we can choose K_n so that $\lim_{n \rightarrow \infty} K_n = \infty$ and $|\partial_{K_n} S_n| \leq \varepsilon n$ for all n . Let S'_n be the set of $\mathbf{x} \in \mathbf{Z}^d$ such that $Q_{1/2}(\mathbf{x})$ has non-empty intersection with $S_n \setminus \partial_{K_n}(S_n)$. Using the conclusion of the previous paragraph and translation invariance, it is not hard to deduce that

$$\lim_{n \rightarrow \infty} \sup_{\mathbf{x} \in S'_n} E[|F_{\mathbf{x}}(R_n; S_n)^2 - F_{\mathbf{x}}^2|] = 0. \quad (75)$$

Also, since we assume $|S_n| \sim n$ we have $\text{card}(S'_n) \geq |S_n| - \varepsilon n \geq (1 - 2\varepsilon)n$ for large enough n . Using this with (75), the uniform boundedness of $E[|F_{\mathbf{x}}(R_n; S_n)^2 - F_{\mathbf{x}}^2|]$ and the fact that ε can be taken arbitrarily small in the above argument, it is routine to deduce that

$$k_n^{-1} \sum_{\mathbf{x} \in R'_n} (F_{\mathbf{x}}(R_n; S_n)^2 - F_{\mathbf{x}}^2) \xrightarrow{L^1} 0,$$

and therefore (72) remains true with $F_{\mathbf{x}}$ replaced by $F_{\mathbf{x}}(R_n; S_n)$; that is, (71) holds and the proof of Theorem 4.2 is complete. \square

4.4 Proof of Theorem 4.1: the non-Poisson case

In this section we complete the proof of Theorem 4.1. The first step is to show that the conditions of Theorem 4.1 imply those of Theorem 4.2, as follows.

Lemma 4.3 *If H satisfies the uniform bounded moments condition (62) and is polynomially bounded, then H satisfies the Poisson bounded moments condition (64).*

Proof. The proof follows, with minor modifications, that of Lemma 4.1 of [17]. \square

It follows from Lemma 4.3 that if H satisfies the conditions of Theorem 4.1, then Theorem 4.2 applies and we have the Poisson parts of Theorem 4.1. To de-Poissonize these limits we follow [17]. Define

$$R_{m,n} := H(\mathcal{U}_{m+1,n}; B) - H(\mathcal{U}_{m,n}; B).$$

We use the following coupling lemma.

Lemma 4.4 *Suppose H is strongly stabilizing. Let $\varepsilon > 0$. Then there exists $\delta > 0$ and $n_0 \geq 1$ such that for all $n \geq n_0$ and all $m, m' \in [(1 - \delta)n, (1 + \delta)n]$ with $m < m'$, there exists a coupled family of variables D, D', R, R' with the following properties:*

- (i) D and D' each have the same distribution as $\Delta(\infty)$;
- (ii) D and D' are independent;
- (iii) (R, R') have the same joint distribution as $(R_{m,n}, R_{m',n})$;
- (iv) $P[\{D \neq R\} \cup \{D' \neq R'\}] < \varepsilon$.

Proof. Since we assume $|S_n|/|R_n| \rightarrow 1$, the probability that a random d -vector uniformly distributed over R_n lies in S_n tends to 1 as $n \rightarrow \infty$. Using this fact the proof follows, with some minor modifications, that of the corresponding result in [17], Lemma 4.2. \square

Lemma 4.5 *Suppose H is strongly stabilizing and satisfies the uniform bounded moments condition (62). Let $(h(n))_{n \geq 1}$ be a sequence with $n^{-1}h(n) \rightarrow 0$ as $n \rightarrow \infty$.*

Then

$$\lim_{n \rightarrow \infty} \sup_{|n-m| \leq h(n)} |ER_{m,n} - E\Delta(\infty)| = 0; \quad (76)$$

$$\lim_{n \rightarrow \infty} \sup_{n-h(n) \leq m < m' \leq n+h(n)} |ER_{m,n}R_{m',n} - (E\Delta(\infty))^2| = 0; \quad (77)$$

$$\lim_{n \rightarrow \infty} \sup_{|n-m| \leq h(n)} ER_{m,n}^2 < \infty. \quad (78)$$

Proof. The proof follows that of Lemma 4.3 of [17]. \square

Proof of Theorem 4.1 Theorem 4.1 now follows in the same way as Theorem 2.1 in [17], replacing $H(\cdot)$ with $H(\cdot; S_n)$. \square

5 Proof of Theorem 2.1: Laws of large numbers

We now derive our law of large numbers for the total weight of the random MDSF on the unit square. We consider the general partial order $\preceq^{\theta, \phi}$, for $0 \leq \theta < 2\pi$ and $0 < \phi \leq \pi$ or $\phi = 2\pi$. Recall that $\mathbf{y} \preceq^{\theta, \phi} \mathbf{x}$ if $\mathbf{y} \in C_{\theta, \phi}(\mathbf{x})$, where $C_{\theta, \phi}(\mathbf{x})$ is the cone formed by the rays at θ and $\theta + \phi$ measured anticlockwise from the upwards vertical.

We consider the random point set \mathcal{X}_n , the binomial point process of n independent uniformly distributed points on $(0, 1]^2$. However, the result (2) also holds (with virtually the same proof) if the points of \mathcal{X}_n are uniformly distributed on an arbitrary convex set in \mathbf{R}^2 of unit area. If the points are distributed in \mathbf{R}^2 with a density function f that has convex support and is bounded away from 0 and infinity on its support, then (2) holds with a factor of $\int_{\mathbf{R}^2} f(\mathbf{x})^{(2-\alpha)/2} d\mathbf{x}$ introduced into the right hand side (cf. eqn (2.9) of [18]).

For the general partial order given by θ, ϕ we apply Lemma 4.1 to obtain a law of large numbers for $\mathcal{L}^\alpha(\mathcal{X}_n)$. As a special case, we thus obtain a law of large numbers under the partial order \preceq^* given by $\theta = \phi = \pi/2$. This method enables us to evaluate the limit explicitly, unlike methods based on the subadditivity of the functional which may also be applicable here (see the remark at the end of this section).

In applying Lemma 4.1 to the MDSF functional, we take the dimension d in the lemma to be 2, and take $f(\mathbf{x})$ (the underlying probability density function in the lemma) to be 1 for $\mathbf{x} \in (0, 1]^2$ and zero elsewhere. We take $\xi(\mathbf{x}; \mathcal{X})$ to be $d(\mathbf{x}; \mathcal{X})^\alpha$, where $d(\mathbf{x}; \mathcal{X})$ is the distance from point \mathbf{x} to its directed nearest neighbour in \mathcal{X} under $\preceq^{\theta, \phi}$, if such a neighbour exists, or zero otherwise. Thus in our case

$$\xi(\mathbf{x}; \mathcal{X}) = (d(\mathbf{x}; \mathcal{X}))^\alpha \quad \text{with} \quad d(\mathbf{x}; \mathcal{X}) := \min \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}, \mathbf{y} \preceq \mathbf{x} \} \quad (79)$$

with the convention that $\min\{\} = 0$. We need to show this choice of ξ satisfies the conditions of Lemma 4.1. As before, \mathcal{H}_τ denotes a homogeneous Poisson process on \mathbf{R}^d of intensity τ , now with $d = 2$.

Lemma 5.1 *Let $\tau > 0$. Then ξ is almost surely stabilizing on \mathcal{H}_τ , in the sense of (57), with limit $\xi_\infty(\mathcal{H}_\tau) = (d(\mathbf{0}; \mathcal{H}_\tau))^\alpha$.*

Proof. Let R be the (random) distance from $\mathbf{0}$ to its directed nearest neighbour in \mathcal{H}_τ , i.e. $R = d(\mathbf{0}; \mathcal{H}_\tau)$. Since $\phi > 0$ and $\tau > 0$, we have $0 < R < \infty$ almost surely. But then for any $\ell > R$, we have $\xi(\mathbf{0}; (\mathcal{H}_\tau \cap B(\mathbf{0}; \ell)) \cup \mathcal{A}) = R^\alpha$, for any finite $\mathcal{A} \subset \mathbf{R}^d \setminus B(\mathbf{0}; \ell)$. Thus ξ stabilizes on \mathcal{H}_τ with limit $\xi_\infty(\mathcal{H}_\tau) = R^\alpha$. \square

Before proving that our choice of ξ satisfies the moments condition for Lemma 4.1, we give a geometrical lemma. For $B \subseteq \mathbf{R}^2$ with B bounded, and for $\mathbf{x} \in B$, write $\text{dist}(\mathbf{x}; \partial B)$ for $\sup\{r : B(\mathbf{x}; r) \subseteq B\}$, and for $s > 0$, define the region

$$A_{\theta, \phi}(\mathbf{x}, s; B) := B(\mathbf{x}; s) \cap B \cap C_{\theta, \phi}(\mathbf{x}). \quad (80)$$

Lemma 5.2 *Let B be a convex bounded set in \mathbf{R}^2 , and let $\mathbf{x} \in B$. If $A_{\theta, \phi}(\mathbf{x}, s; B) \cap \partial B(\mathbf{x}; s) \neq \emptyset$, and $s > \text{dist}(\mathbf{x}, \partial B)$, then*

$$|A_{\theta, \phi}(\mathbf{x}, s; B)| \geq s \sin(\phi/2) \text{dist}(\mathbf{x}, \partial B)/2.$$

Proof. The condition $A_{\theta, \phi}(\mathbf{x}, s; B) \cap \partial B(\mathbf{x}; s) \neq \emptyset$ says that there exists $\mathbf{y} \in B \cap C_{\theta, \phi}(\mathbf{x}, s)$ with $\|\mathbf{y} - \mathbf{x}\| = s$. The line segment $\mathbf{x}\mathbf{y}$ is contained in the cone $C_{\theta, \phi}(\mathbf{x})$; take a half-line \mathbf{h} starting from \mathbf{x} , at an angle $\phi/2$ to the line segment $\mathbf{x}\mathbf{y}$ and such that \mathbf{h} is also contained in $C_{\theta, \phi}(\mathbf{x})$. Let \mathbf{z} be the point in \mathbf{h} at a distance $\text{dist}(\mathbf{x}, \partial B)$ from \mathbf{x} . Then the interior of the triangle $\mathbf{x}\mathbf{y}\mathbf{z}$ is entirely contained in $A_{\theta, \phi}(\mathbf{x}, s)$, and has area $s \sin(\phi/2) \text{dist}(\mathbf{x}, \partial B)/2$. \square

Lemma 5.3 *Suppose $\alpha > 0$. Then ξ given by (79) satisfies the moments condition (58) for any $p \in (1/\alpha, 2/\alpha]$.*

Proof. Setting $R_n := (0, n^{1/2}]^2$, we have

$$E \left[\xi \left(n^{1/2} \mathbf{X}_1; n^{1/2} \mathcal{X}_n \right)^p \right] = \int_{R_n} E \left[\left(\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) \right)^p \right] \frac{d\mathbf{x}}{n}. \quad (81)$$

For $\mathbf{x} \in R_n$ set $m(\mathbf{x}) := \text{dist}(\mathbf{x}, \partial R_n)$. Let us divide R_n into three regions

$$\begin{aligned} R_n(1) &:= \{\mathbf{x} \in R_n : m(\mathbf{x}) \leq n^{-1/2}\}; & R_n(2) &:= \{\mathbf{x} \in R_n : m(\mathbf{x}) > 1\}; \\ R_n(3) &:= \{\mathbf{x} \in R_n : n^{-1/2} < m(\mathbf{x}) \leq 1\}. \end{aligned}$$

For all $\mathbf{x} \in R_n$, we have $\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) \leq (2n)^{\alpha/2}$, and hence, since $R_n(1)$ has area at most 4, we can bound the contribution to (81) from $\mathbf{x} \in R_n(1)$ by

$$\int_{\mathbf{x} \in R_n(1)} E \left[\left(\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) \right)^p \right] \frac{d\mathbf{x}}{n} \leq 4n^{-1} (2n)^{p\alpha/2} = 2^{2+p\alpha/2} n^{(p\alpha-2)/2}, \quad (82)$$

which is bounded provided $p\alpha \leq 2$.

Now, for $\mathbf{x} \in R_n$, with $A_{\theta,\phi}(\cdot)$ defined at (80), we have

$$\begin{aligned} P \left[d(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) > s \right] &\leq P \left[n^{1/2} \mathcal{X}_{n-1} \cap A_{\theta,\phi}(\mathbf{x}, s; R_n) = \emptyset \right] \\ &= \left(1 - \frac{|A_{\theta,\phi}(\mathbf{x}, s; R_n)|}{n} \right)^{n-1} \\ &\leq \exp(1 - |A_{\theta,\phi}(\mathbf{x}, s; R_n)|), \end{aligned} \quad (83)$$

since $|A_{\theta,\phi}(\mathbf{x}, s; R_n)| \leq n$. For $\mathbf{x} \in R_n$ and $s > m(\mathbf{x})$, by Lemma 5.2 we have

$$|A_{\theta,\phi}(\mathbf{x}, s; R_n)| \geq \sin(\phi/2)sm(\mathbf{x})/2 \quad \text{if } A_{\theta,\phi}(\mathbf{x}, s; R_n) \cap \partial B(\mathbf{x}; s) \neq \emptyset,$$

and also

$$P[d(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) > s] = 0 \quad \text{if } A_{\theta,\phi}(\mathbf{x}, s; R_n) \cap \partial B(\mathbf{x}; s) = \emptyset.$$

For $s \leq m(\mathbf{x})$, we have that $|A_{\theta,\phi}(\mathbf{x}, s; R_n)| = \frac{\phi}{2}s^2 \geq \sin(\phi/2)s^2$. Combining these observations and (83), we obtain for all $\mathbf{x} \in R_n$ and $s > 0$ that

$$P \left[d(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) > s \right] \leq \exp(1 - \sin(\phi/2)s \min(s, m(\mathbf{x}))/2), \quad \mathbf{x} \in R_n.$$

Setting $c = (1/2)\sin(\phi/2)$, we therefore have for $\mathbf{x} \in R_n$ that

$$\begin{aligned} E \left[\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1})^p \right] &= \int_0^\infty P \left[\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1})^p > r \right] dr \\ &= \int_0^\infty P \left[d(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) > r^{1/(\alpha p)} \right] dr \\ &\leq \int_0^{m(\mathbf{x})^{\alpha p}} dr \exp(1 - cr^{2/(\alpha p)}) \\ &\quad + \int_{m(\mathbf{x})^{\alpha p}}^\infty dr \exp(1 - cm(\mathbf{x})r^{1/(\alpha p)}) \\ &= O(1) + \int_{m(\mathbf{x})^2}^\infty e^{1-cu} \alpha p u^{\alpha p-1} m(\mathbf{x})^{-p\alpha} du \\ &= O(1) + O(m(\mathbf{x})^{-\alpha p}). \end{aligned} \quad (84)$$

For $\mathbf{x} \in R_n(2)$, this bound is $O(1)$, and the area of $R_n(2)$ is less than n , so that the contribution to (81) from $R_n(2)$ satisfies

$$\limsup_{n \rightarrow \infty} \int_{R_n(2)} E \left[\left(\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) \right)^p \right] \frac{d\mathbf{x}}{n} < \infty. \quad (85)$$

Finally, by (84), there is a constant c' such that if $\alpha p > 1$, the contribution to (81) from $R_n(3)$ satisfies

$$\begin{aligned} \int_{R_n(3)} E \left[\left(\xi(\mathbf{x}; n^{1/2} \mathcal{X}_{n-1}) \right)^p \right] \frac{d\mathbf{x}}{n} &\leq c' n^{-1/2} \int_{y=n^{-1/2}}^1 y^{-\alpha p} dy \\ &\leq \left(\frac{c' n^{-1/2}}{\alpha p - 1} \right) n^{(\alpha p - 1)/2} \end{aligned}$$

which is bounded provided $\alpha p \leq 2$. Combined with the bounds in (82) and (85), this shows that the expression (81) is uniformly bounded, provided $1 < \alpha p \leq 2$. \square

Following notation from Section 4.2, for $k \in \mathbf{N}$, and for $a < b$ and $c < d$ let $\mathcal{U}_{k,(a,b] \times (c,d]}$ denote the point process consisting of k independent random vectors uniformly distributed on the rectangle $(a, b] \times (c, d]$. Before proceeding further, we recall that if $M(\mathcal{X})$ denotes the number of minimal elements (under the ordering \preceq^*) of a point set $\mathcal{X} \subset \mathbf{R}^2$, then

$$E[M(\mathcal{U}_{k,(a,b] \times (c,d]})] = E[M(\mathcal{X}_k)] = 1 + (1/2) + \cdots + (1/k) \leq 1 + \log k. \quad (86)$$

The first equality in (86) comes from some obvious scaling which shows that the distribution of $M(\mathcal{U}_{k,(a,b] \times (c,d]})$ does not depend on a, b, c, d . For the second equality in (86), see [3] or the proof of Theorem 1.1(a) of [6].

Proof of Theorem 2.1. Suppose $\alpha < 2$, and set $f(\cdot)$ to be the indicator of the unit square $(0, 1]^2$. By Lemmas 5.1 and 5.3, our functional ξ , given at (79), satisfies the conditions of Lemma 4.1 with $p = 2/\alpha$ and $q = 1$, with this choice of f . So by Lemma 4.1, we have that

$$\begin{aligned} n^{(\alpha/2)-1} \mathcal{L}^\alpha(\mathcal{X}_n) &= n^{-1} \sum_{\mathbf{x} \in \mathcal{X}_n} \xi(n^{1/2}\mathbf{x}; n^{1/2}\mathcal{X}_n) \\ &\xrightarrow{L^1} \int_{\mathbf{R}^2} E[\xi_\infty(\mathcal{H}_{f(\mathbf{x})})] f(\mathbf{x}) d\mathbf{x} = E\xi_\infty(\mathcal{H}_1). \end{aligned} \quad (87)$$

Since the disk sector $C_{\theta,\phi}(\mathbf{x}) \cap B(\mathbf{x}; r)$ has area $(\phi/2)r^2$, by Lemma 5.1 we have

$$P[\xi_\infty(\mathcal{H}_1) > s] = P[\mathcal{H}_1 \cap C_{\theta,\phi}(\mathbf{0}) \cap B(\mathbf{0}; s^{1/\alpha}) = \emptyset] = \exp\left(-(\phi/2)s^{2/\alpha}\right).$$

Hence, the limit in (87) is

$$E[\xi_\infty(\mathcal{H}_1)] = \int_0^\infty P[\xi_\infty(\mathcal{H}_1) > s] ds = \alpha 2^{(\alpha-2)/2} \phi^{-\alpha/2} \Gamma(\alpha/2),$$

and this gives us (2). Finally, in the case where $\preceq^{\theta,\phi} = \preceq^*$, (2) remains true when \mathcal{X}_n is replaced by \mathcal{X}_n^0 , since

$$E[n^{(\alpha/2)-1} |\mathcal{L}^\alpha(\mathcal{X}_n^0) - \mathcal{L}^\alpha(\mathcal{X}_n)|] \leq 2^{\alpha/2} n^{(\alpha/2)-1} E[M(\mathcal{X}_n)], \quad (88)$$

where $M(\mathcal{X}_n)$ denotes the number of minimal elements of \mathcal{X}_n . By (86), $E[M(\mathcal{X}_n)] \leq 1 + \log n$, and hence the right hand side of (88) tends to 0 as $n \rightarrow \infty$ for $0 < \alpha < 2$. This gives us (2) with \mathcal{X}_n^0 under \preceq^* . \square

Remark. A law of large numbers for Euclidean functionals of many random geometric structures can be treated by the boundary functional approach of Yukich [25]. It can be shown that the MDSF satisfies some, but possibly not all, of the appropriate conditions that would allow this approach to be successful. The MDSF functional is subadditive, its corresponding boundary functional is superadditive, and

the functional and its boundary functional are sufficiently ‘close in mean’. However, it is not clear that the functional is ‘smooth’, since the degree of the graph is not bounded.

6 Central limit theorem away from the boundary

While it should be possible to adapt the argument of the present section to more general partial orders, from now on we take the partial order \preceq on \mathbf{R}^2 to be \preceq^* . For each n , define the region $S_{0,n} := (n^{\varepsilon-1/2}, 1]^2$, where $\varepsilon \in (0, 1/2)$ is a small constant to be chosen later. In this section, we use the general central limit theorems of Section 4.2 to demonstrate a central limit theorem for the contribution to the total weight of the MDSF, under \preceq^* , from edges away from the boundary, that is from points in the region $S_{0,n}$.

Given $\alpha > 0$, consider the MDSF total weight functional $H = \mathcal{L}^\alpha$ on point sets in \mathbf{R}^2 . For $\mathbf{x} \in \mathcal{X}$, let the directed nearest neighbour distance $d(\mathbf{x}; \mathcal{X})$ and the corresponding α -weighted functional $\xi(\mathbf{x}; \mathcal{X})$ be given by (79), where now we take \preceq to be \preceq^* . For $R \subseteq \mathbf{R}^2$, set

$$\mathcal{L}^\alpha(\mathcal{X}; R) = \sum_{\mathbf{x} \in \mathcal{X} \cap R} \xi(\mathbf{x}; \mathcal{X}), \quad (89)$$

and set $\mathcal{L}^\alpha(\mathcal{X}) := \mathcal{L}^\alpha(\mathcal{X}; \mathbf{R}^2)$.

Let \mathcal{X}_n be the binomial point process of n i.i.d. uniform random vectors on $(0, 1]^2$, and let \mathcal{P}_n be the homogeneous Poisson process of intensity n on $(0, 1]^2$. The main result of this section is the following.

Theorem 6.1 *Suppose that $\alpha > 0$ and the partial order is \preceq^* . Then there exist constants $0 < t_\alpha \leq s_\alpha$, not depending on the choice of ε , such that, as $n \rightarrow \infty$,*

- (i) $n^{\alpha-1} \text{Var} [\mathcal{L}^\alpha(\mathcal{X}_n; S_{0,n})] \rightarrow t_\alpha^2$;
- (ii) $n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; S_{0,n}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, t_\alpha^2)$;
- (iii) $n^{\alpha-1} \text{Var} [\mathcal{L}^\alpha(\mathcal{P}_n; S_{0,n})] \rightarrow s_\alpha^2$;
- (iv) $n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; S_{0,n}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s_\alpha^2)$.

The following corollary states that Theorem 6.1 remains true in the rooted cases too, i.e. with \mathcal{X}_n replaced by \mathcal{X}_n^0 and \mathcal{P}_n replaced by \mathcal{P}_n^0 .

Corollary 6.1 *Suppose that $\alpha > 0$ and the partial order is \preceq^* . Then, with t_α, s_α as given in Theorem 6.1, we have that as $n \rightarrow \infty$,*

- (i) $n^{\alpha-1} \text{Var} [\mathcal{L}^\alpha(\mathcal{X}_n^0; S_{0,n})] \rightarrow t_\alpha^2$;
- (ii) $n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n^0; S_{0,n}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, t_\alpha^2)$;
- (iii) $n^{\alpha-1} \text{Var} [\mathcal{L}^\alpha(\mathcal{P}_n^0; S_{0,n})] \rightarrow s_\alpha^2$;

$$(iv) \quad n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n^0; S_{0,n}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s_\alpha^2).$$

Proof. For each region $R \subseteq [0, 1]^2$ and point set $\mathcal{S} \subset [0, 1]^2$ with $\mathbf{0} \in \mathcal{S}$, let $\mathcal{L}_0^\alpha(\mathcal{S}; R)$ denote the total weight of the edges incident to $\mathbf{0}$ in the MDST on \mathcal{S} from points in R . Then $\mathcal{L}^\alpha(\mathcal{P}_n^0; S_{0,n})$ equals $\mathcal{L}^\alpha(\mathcal{P}_n; S_{0,n}) + \mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})$, so that

$$\begin{aligned} \text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n^0; S_{0,n})] - \text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n; S_{0,n})] &= 2\text{Cov}[\mathcal{L}^\alpha(\mathcal{P}_n; S_{0,n}), \mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})] \\ &\quad + \text{Var}[\mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})]. \end{aligned} \quad (90)$$

Let N_n denote the number of points of \mathcal{P}_n , and let E_n denote the event that at least one point of $\mathcal{P}_n \cap S_{0,n}$ is joined to $\mathbf{0}$ in the MDST on \mathcal{P}_n^0 . Then

$$P[E_n] \leq P\left[(0, n^{\varepsilon-1/2}]^2 \cap \mathcal{P}_n = \emptyset\right] = \exp(-n^{2\varepsilon}),$$

and $\mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n}) \leq 2^{\alpha/2} N_n \mathbf{1}_{E_n}$. Thus by the Cauchy-Schwarz inequality, for some finite constant C we have

$$\text{Var}[\mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})] \leq E[\mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})^2] \leq Cn^2 \exp(-n^{2\varepsilon}/2), \quad (91)$$

and combining this with (90), Theorem 6.1 (iii) and the Cauchy-Schwarz inequality shows that

$$n^{\alpha-1}(\text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n^0; S_{0,n})] - \text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n; S_{0,n})]) \rightarrow 0,$$

so that from Theorem 6.1 (iii) we obtain the corresponding rooted result (iii). Also, since (91) implies $n^{\alpha-1} \text{Var}[\mathcal{L}_0^\alpha(\mathcal{P}_n^0; S_{0,n})]$ tends to zero, from Theorem 6.1 (iv) and Slutsky's theorem we obtain the corresponding rooted result (iv).

The binomial results (i) and (ii) follow in the same manner as above, with slight modifications. \square

To prove Theorem 6.1, we demonstrate that our functional \mathcal{L}^α satisfies suitable versions of the conditions of Theorem 4.1 and Corollary 4.1. First, we see that \mathcal{L}^α is polynomially bounded (see (61)), since

$$\mathcal{L}^\alpha(\mathcal{X}; B) \leq (\text{diam}(\mathcal{X}))^\alpha \text{card}(\mathcal{X}).$$

Also, \mathcal{L}^α is homogeneous of order α .

Lemma 6.1 *\mathcal{L}^α is strongly stabilizing, in the sense of Definition 4.1.*

Proof. To prove stabilization it is sufficient to show that there exists an almost surely finite random variable R , the radius of stabilization, such that the add one cost is unaffected by changes in the configuration at a distance greater than R from the added point. We show that there exists such an R .

For $s > 0$ construct eight disjoint triangles $T_j(s)$, $1 \leq j \leq 8$, by splitting the square $Q(\mathbf{0}; s)$ into eight triangles via drawing in the diagonals of the square and the x and y axes. Label the triangle with vertices $(0, 0)$, $(0, s)$, (s, s) as $T_1(s)$ and then label increasingly in a clockwise manner. See Figure 3. Note that $T_j(t) \subset T_j(s)$

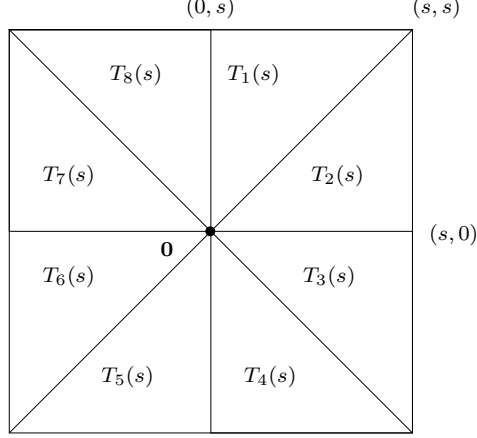


Figure 3: The triangles $T_1(s), \dots, T_8(s)$, $s > 0$.

for $t < s$. Let the random variable S be the minimum s such that the triangles $T_j(s)$, $1 \leq j \leq 8$, each contain at least one point of \mathcal{P} . Then S is almost surely finite.

We claim that $R = 3S$ is a radius of stabilization for \mathcal{L}^α , that is any points at distance $d \geq 3S$ from the origin have no impact on the set of added or removed edges when a point is inserted at the origin.

First, $\mathbf{0}$ can have no point at a distance of at least $3S$ away as its directed nearest neighbour, since there will be points in T_5 and T_6 within a distance of at most $\sqrt{2}S$ of $\mathbf{0}$.

We now need to show that no point at a distance at least $3S$ from $\mathbf{0}$ can have the origin as its directed nearest neighbour. Clearly, for the partial order \preceq^* , we need only consider points in the region $(0, \infty)^2$.

Consider a point (x, y) in the first quadrant, such that $\|(x, y)\| \geq 3S$. Consider the disk sector

$$D_{(x,y)} := B((x, y), \|(x, y)\|) \cap \{\mathbf{w} : \mathbf{w} \preceq^*(x, y)\}.$$

We aim to show that given any (x, y) of the above form, at least one of the $T_j(S)$, $j = 1, \dots, 8$, is contained in $D_{(x,y)}$, which implies that the origin cannot be the directed nearest neighbour of (x, y) . To demonstrate this, we show that given such an (x, y) , $D_{(x,y)}$ contains all three vertices of at least one of the $T_j(S)$.

First suppose $x > S$, $y > S$. Then we have that $T_1(S)$ and $T_2(S)$ are in $D_{(x,y)}$, since we have, for example,

$$\begin{aligned} \|(x, y) - \mathbf{0}\|^2 - \|(x, y) - (0, S)\|^2 &= (x^2 + y^2) - (x^2 + (y - S)^2) \\ &= S(2y - S) > 0. \end{aligned}$$

By symmetry, the only other situation we need consider is when $0 < x \leq S$. Then $y^2 \geq 9S^2 - x^2 \geq 8S^2$, so $y \geq 2\sqrt{2}S$. Then we have that $T_8(S)$ is in $D_{(x,y)}$, since

$$\|(x, y) - \mathbf{0}\|^2 - \|(x, y) - (-S, S)\|^2 = (x^2 + y^2) - ((x + S)^2 + (y - S)^2)$$

$$= 2S(y - x - S) \geq 4S^2(\sqrt{2} - 1) > 0.$$

This completes the proof. \square

Lemma 6.2 *The distribution of $\Delta(\infty)$ is non-degenerate.*

Proof. We demonstrate the existence of two configurations that occur with strictly positive probability and give rise to different values for $\Delta(\infty)$. Note that adding a point at the origin causes some new edges to be formed (namely those incident to the origin), and the possible deletion of some edges (namely the edges from points which have the origin as their directed nearest neighbour after its insertion).

Let $\eta > 0$, with $\eta < 1/3$. Later we shall impose further conditions on η . Again we refer to the construction in Figure 3. Let E_1 denote the event that for each i , $1 \leq i \leq 8$, there is a single point of \mathcal{P} , denoted \mathbf{W}_i , in each of $T_i(\eta)$, and that there are no other points in $[-1, 1]^2$. Suppose that E_1 occurs. Then, on addition of the origin, the only edges that can possibly be removed are those from \mathbf{W}_1 and from \mathbf{W}_2 (see the proof of Lemma 6.1). These removed edges have length at most $\eta\sqrt{8}$, and hence

$$\Delta \geq -2(\eta\sqrt{8})^\alpha := \delta_1, \quad \text{on } E_1. \quad (92)$$

Now let E_2 denote the event that there is a single point of \mathcal{P} , denoted \mathbf{Z}_1 , in the square $(\eta, 2\eta) \times (0, \eta)$, a single point denoted \mathbf{Z}_2 in the square $(0, \eta) \times (\eta, 2\eta)$, a single point denoted \mathbf{W} in the square $(-1-\eta, -1) \times (-\eta, 0)$, and no other point in $[-3, 3]^2$. See Figure 4.

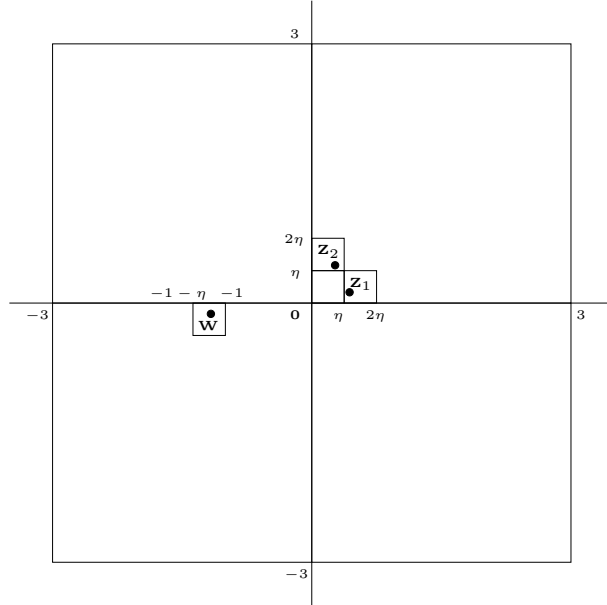


Figure 4: A possible configuration for event E_2 .

Suppose that E_2 occurs. Now, on addition of the origin, an edge of length at most $1 + 2\eta$ is added from the origin to \mathbf{W} . On the other hand, for $i = 1, 2$ the edge from \mathbf{Z}_i to \mathbf{W} (of length at least 1) is replaced by an edge from \mathbf{Z}_i to the origin (of length at most 3η). It is also possible that some other edges from points outside $[-3, 3]^2$ are replaced by shorter edges from these points to the origin. Combining the effect of all these additions and replacements of edges, we find that

$$\Delta \leq (1 + 2\eta)^\alpha + 2((3\eta)^\alpha - 1) := \delta_2, \quad \text{on } E_2. \quad (93)$$

Given α , by taking η small enough we can arrange that $\delta_1 > -1/4$ and $\delta_2 < -3/4$. With such a choice of η , events E_1 and E_2 both have strictly positive probability which shows that the distribution of Δ is non-degenerate. \square

For the next lemma, we set $R_0 := (0, 1]^2$, recalling that $S_{0,n} := (n^{\varepsilon-1/2}, 1]^2$ throughout this section, and let \mathcal{R}_0 be as defined just before Corollary 4.1.

Lemma 6.3 \mathcal{L}^α satisfies the uniform bounded moments condition (62) on \mathcal{R}_0 .

Proof. Choose some $(A, B) \in \mathcal{R}_0$ such that $\mathbf{0} \in A$, i.e., such that for some $n \in \mathbb{N}$ the set A is a translate of $(0, n^{1/2}]^2$ containing the origin and B is the corresponding translate of $n^{1/2}S_{0,n} = (n^\varepsilon, n^{1/2}]^2$. Note that $|A| = n$, and choose $m \in [n/2, 3n/2]$.

Denote the m independent random vectors on A comprising $\mathcal{U}_{m,A}$ by $\mathbf{V}_1, \dots, \mathbf{V}_m$. For contributions to $\Delta(\mathcal{U}_{m,A}; B)$ we are only interested in edges from points in the region B away from the boundary of A , although the origin can be inserted anywhere in A . Contributions to $\Delta(\mathcal{U}_{m,A}; B)$ come from the edges that are added or deleted on the addition of $\mathbf{0}$. We split $\Delta(\mathcal{U}_{m,A}; B)$ into two parts: the positive contribution from added edges, $\Delta^+(\mathcal{U}_{m,A}; B)$, and the negative contribution, $\Delta^-(\mathcal{U}_{m,A}; B)$, from removed edges.

By construction of the MDSF, the added edges are those that have $\mathbf{0}$ as an end-point after it has been inserted. Thus an upper bound on $\Delta^+(\mathcal{U}_{m,A}; B)$ is $L_{\max}^\alpha \delta(\mathbf{0}) + L_0^\alpha$, where L_{\max} is the length of the longest edge from a point of $\mathcal{U}_{m,A} \cap B$ to $\mathbf{0}$, and $\delta(\mathbf{0})$ is the number of such edges (or zero if no such edge exists), and L_0 is the length of the edge from $\mathbf{0}$, or zero if no such edge exists.

For $\mathbf{w} \in A$ and $\mathbf{x} \in B$, with $\mathbf{w} \preceq^* \mathbf{x}$, define the region

$$R(\mathbf{w}, \mathbf{x}) := \{\mathbf{y} \in A : \mathbf{y} \preceq^* \mathbf{x}, \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{w} - \mathbf{x}\|\}.$$

Since points in B are distant at least 1 from the lower or left boundary of A , by Lemma 5.2 there exists a constant $0 < C < \infty$ such that

$$|R(\mathbf{w}, \mathbf{x})| \geq C\|\mathbf{x} - \mathbf{w}\|, \text{ for all } \mathbf{w} \in A, \mathbf{x} \in B \text{ with } \mathbf{w} \preceq^* \mathbf{x} \text{ and } \|\mathbf{x} - \mathbf{w}\| \geq 1. \quad (94)$$

Suppose there is a point at \mathbf{x} with $\mathbf{0} \preceq^* \mathbf{x}$. Then, the probability of the event $E(\mathbf{x})$ that \mathbf{x} is joined to the origin in the MDSF on $\mathcal{U}_{m,A} \cup \{\mathbf{0}\}$ is

$$\begin{aligned} P[E(\mathbf{x})] &= P[R(\mathbf{0}, \mathbf{x}) \text{ empty}] = \left(1 - \frac{|R(\mathbf{0}, \mathbf{x})|}{|A|}\right)^{m-1} \\ &\leq \exp\left((1-m)\left(\frac{|R(\mathbf{0}, \mathbf{x})|}{n}\right)\right) \leq \exp(1 - |R(\mathbf{0}, \mathbf{x})|/2), \end{aligned} \quad (95)$$

since $m \geq n/2$ and $|R(\mathbf{0}, \mathbf{x})| \leq n$.

We have that $L_{\max}^\alpha \delta(\mathbf{0}) \leq \max_{i=1, \dots, m} W_i$, where

$$W_i = \|\mathbf{V}_i\|^\alpha \text{card}(B(\mathbf{0}; \|\mathbf{V}_i\|) \cap \mathcal{U}_{m,A} \cap \{\mathbf{y} : \mathbf{0} \preceq^* \mathbf{y}\}) \mathbf{1}\{\mathbf{V}_i \text{ joined to } \mathbf{0} \text{ and } \mathbf{V}_i \in B\}.$$

Let $N(\mathbf{x})$ denote the number of points of $\mathcal{U}_{m-1,A}$ in $B(\mathbf{0}; \|\mathbf{x}\|) \cap \{\mathbf{y} : \mathbf{0} \preceq \mathbf{y}\}$. Then we obtain

$$E[L_{\max}^{4\alpha} \delta(\mathbf{0})^4] \leq E \sum_{i=1}^m W_i^4 = m \int_B \|\mathbf{x}\|^{4\alpha} E[(N(\mathbf{x}) + 1)^4 \mathbf{1}\{E(\mathbf{x})\}] \frac{d\mathbf{x}}{|A|}.$$

By the Cauchy-Schwarz inequality and the fact that $m \leq 3|A|/2$ by assumption,

$$E[L_{\max}^{4\alpha} \delta(\mathbf{0})^4] \leq \frac{3}{2} \int_B \|\mathbf{x}\|^{4\alpha} (E[(N(\mathbf{x}) + 1)^8])^{1/2} P[E(\mathbf{x})]^{1/2} d\mathbf{x}. \quad (96)$$

The mean of $N(\mathbf{x})$ is bounded by a constant times $\|\mathbf{x}\|^2$ so $E[(N(\mathbf{x}) + 1)^8] = O(\max(\|\mathbf{x}\|^{16}, 1))$. This follows from the binomial moment generating function for $\text{Bin}(n, p)$, from which we have for $\beta > 0$ that $E[X^\beta] \leq k_1(E[X])^\beta$ if $pn > 1$ and $E[X^\beta] \leq k_2 E[X]$ if $pn < 1$, for some constants $k_1, k_2 > 0$.

Combined with (94), (95) and (96), this shows that $E[L_{\max}^{4\alpha} \delta(\mathbf{0})^4]$ is bounded by a constant times

$$\int_{\mathbf{x} \in B: \|\mathbf{x}\| \geq 1} \|\mathbf{x}\|^{4\alpha+8} \exp(-C\|\mathbf{x}\|/4) d\mathbf{x} + \int_{\mathbf{x} \in B: \|\mathbf{x}\| \leq 1} \|\mathbf{x}\|^{4\alpha} d\mathbf{x},$$

which is bounded by a constant that does not depend on the choice of (A, B) .

We need to consider L_0 only when $\mathbf{0} \in B$. For $\mathbf{x} \in \mathbf{R}^2$ with $\mathbf{x} \preceq^* \mathbf{0}$, let $E'(\mathbf{x})$ denote the event that $R(\mathbf{x}, \mathbf{0})$ is empty (i.e., contains no point of $\mathcal{U}_{m-1,A}$). By (94) and (95), for $\mathbf{0} \in B$ we have

$$\begin{aligned} E[L_0^{4\alpha}] &\leq m \int_{\mathbf{x} \in A: \mathbf{x} \preceq^* \mathbf{0}} \|\mathbf{x}\|^{4\alpha} P[E'(\mathbf{x})] \frac{d\mathbf{x}}{|A|} \\ &\leq \frac{3}{2} \left[\int_{\mathbf{x} \in A: \mathbf{x} \preceq^* \mathbf{0}, \|\mathbf{x}\| \geq 1} \|\mathbf{x}\|^{4\alpha} \exp(1 - C\|\mathbf{x}\|/2) d\mathbf{x} + \int_{\mathbf{x} \in A: \mathbf{x} \preceq^* \mathbf{0}, \|\mathbf{x}\| \leq 1} \|\mathbf{x}\|^{4\alpha} d\mathbf{x} \right] \end{aligned}$$

which is bounded by a constant. Thus $\Delta^+(\mathcal{U}_{m,A}; B)$ has bounded fourth moment.

Now consider the set of deleted edges. As at (79), let $d(\mathbf{x}; \mathcal{U}_{m,A})$ denote the distance from \mathbf{x} to its directed nearest neighbour in $\mathcal{U}_{m,A}$, or zero if no such point exists. Again use $E(\mathbf{x})$ for the event that \mathbf{x} becomes joined to $\mathbf{0}$ on the addition of the origin, and let $E''(\mathbf{V}_i) := E(\mathbf{V}_i) \cap \{\mathbf{V}_i \in B\}$. Then

$$\begin{aligned} E[\Delta^-(\mathcal{U}_{m,A}; B)^4] &= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{\ell=1}^m E[d(\mathbf{V}_i; \mathcal{U}_{m,A})^\alpha d(\mathbf{V}_j; \mathcal{U}_{m,A})^\alpha \\ &\quad \times d(\mathbf{V}_k; \mathcal{U}_{m,A})^\alpha d(\mathbf{V}_\ell; \mathcal{U}_{m,A})^\alpha \mathbf{1}\{E''(\mathbf{V}_i) \cap E''(\mathbf{V}_j) \cap E''(\mathbf{V}_k) \cap E''(\mathbf{V}_\ell)\}]. \quad (97) \end{aligned}$$

For i, j, k, ℓ distinct, the (i, j, k, ℓ) th term of (97) is bounded by

$$\int_B \int_B \int_B \int_B \frac{d\mathbf{w}}{n} \frac{d\mathbf{x}}{n} \frac{d\mathbf{y}}{n} \frac{d\mathbf{z}}{n} E[d_{m-4}(\mathbf{w})^\alpha d_{m-4}(\mathbf{x})^\alpha d_{m-4}(\mathbf{y})^\alpha d_{m-4}(\mathbf{z})^\alpha \times \mathbf{1}\{E_{m-4}(\mathbf{w}) \cap E_{m-4}(\mathbf{x}) \cap E_{m-4}(\mathbf{y}) \cap E_{m-4}(\mathbf{z})\}], \quad (98)$$

where $d_{m-4}(\mathbf{x}) := d(\mathbf{x}, \mathcal{U}_{m-4,A} \cup \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\})$ (using the notation of (79)), and $E_{m-4}(\mathbf{x})$ is the event that $\mathbf{0}$ is the directed nearest neighbour of \mathbf{x} in the set $\mathcal{U}_{m-4,A} \cup \{\mathbf{0}, \mathbf{x}\}$.

Let $I_{m-4}(\mathbf{x})$ denote the indicator variable of the event that \mathbf{x} is a minimal element of $\mathcal{U}_{m-4,A} \cup \{\mathbf{x}\}$. An upper bound for $d_{m-4}(\mathbf{x})$ is provided by $d(\mathbf{x}; \mathcal{U}_{m-4,A} \cup \mathbf{x})$ except when this is zero, so that

$$d_{m-4}(\mathbf{x})^{8\alpha} \leq d(\mathbf{x}; \mathcal{U}_{m-4,A} \cup \{\mathbf{x}\})^{8\alpha} + d(\mathbf{x}; \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\})^{8\alpha} I_{m-4}(\mathbf{x}). \quad (99)$$

For $\mathbf{x} \in B$, it can be shown, by a similar argument to the one used above for L_0 , that there is a constant C' such that

$$E[(d(\mathbf{x}; \mathcal{U}_{m-4,A} \cup \{\mathbf{x}\}))^{8\alpha}] < C'. \quad (100)$$

Moreover, if $\mathbf{w} \in A$ with $\mathbf{w} \preceq \mathbf{x}$ and $\|\mathbf{x} - \mathbf{w}\| = t > 0$, then by a similar argument to that at (95), and (94), we have that

$$E[I_{m-4}(\mathbf{x})] \leq \exp(4 - |R(\mathbf{w}, \mathbf{x})|/2) \leq \exp(4 - Ct/2), \quad t \geq 1,$$

and hence, uniformly over A, B and $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\} \subset A$ with $\mathbf{x} \in B$, we have

$$E[d(\mathbf{x}; \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\})^{8\alpha} I_{m-4}(\mathbf{x})] \leq \max \left\{ \sup_{t \geq 1} (t^{8\alpha} \exp(4 - Ct/2)), 1 \right\}.$$

Combining this with (100), we see from (99) that $E[d_{m-4}(\mathbf{x})^{8\alpha}]$ is bounded by a constant. Also, by a similar argument to (95) and (94), it can be shown that $P[E_{m-4}(\mathbf{x})] \leq \exp(4 - C\|\mathbf{x}\|/2)$ for $\|\mathbf{x}\| \geq 1$. Therefore, by Hölder's inequality, the expression (98) is bounded by a constant times

$$n^{-4} \int \int \int \int d\mathbf{w} d\mathbf{x} d\mathbf{y} d\mathbf{z} \exp(-C(\|\mathbf{w}\| + \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{z}\|)/16)$$

and therefore is $O(n^{-4})$. Since the number of distinct (i, j, k, ℓ) in the summation (97) is bounded by m^4 , and hence by $(3/2)^4 n^4$, this shows that the contribution to (97) from i, j, k, ℓ distinct is uniformly bounded.

Likewise, the number of terms (i, j, k, ℓ) with only three distinct values (e.g., $i = j$ with i, k, ℓ distinct) is $O(n^3)$. Such a term is bounded by an expression like (98) but now with a triple integral, which by a similar argument is $O(n^{-3})$. Hence the contribution to (97) of these terms is also bounded. Similarly, the contribution to (97) from (i, j, k, ℓ) with two distinct values has $O(n^2)$ terms which are $O(n^{-2})$, and so is bounded. Likewise the contribution to (97) from terms with $i = j = k = \ell$ is bounded. Thus the expression (97) is uniformly bounded.

Hence $\Delta(\mathcal{U}_{m,A}; B)$ has bounded fourth moments, uniformly in A, B, m . \square

Proof of Theorem 6.1. By Lemmas 6.1, 6.2, 6.3 and the fact that \mathcal{L}^α is homogeneous of order α , we can apply Corollary 4.1, taking $R_0 := (0, 1]^2$ and $S_{0,n} := (n^{\varepsilon-1/2}, 1]^2$, to obtain Theorem 6.1. \square

Remark. An alternative method for proving central limit theorems in geometrical probability is based on dependency graphs. Such a method was employed by Avram and Bertsimas [1] to give central limit theorems for nearest neighbour graphs and other random geometrical structures. A general version of this method is provided by [19]. By a similar argument to [1], one can show that, under \preceq^* , the total weight (for $\alpha > 2/3$) of edges in the MDST from points in the region $(\varepsilon_n, 1)^2$ (for ε_n given below) satisfies a central limit theorem, where

$$\varepsilon_n = \left(\left\lfloor \sqrt{\frac{n}{c \log n}} \right\rfloor \right)^{-1}.$$

Such an approach can be suitably adapted to show that a central limit theorem also holds under the more general partial order specified by θ, ϕ , in the region $(\varepsilon_n, 1 - \varepsilon_n)^2$. The benefit of this method is that it readily yields rates of convergence bounds for the CLT. The martingale method employed has the advantage of yielding the convergence of the variance.

7 The edges near the boundary

Next in our analysis of the MDST on random points in the unit square, we consider the length of the edges close to the boundary of the square. The limiting structure of the MDSF and MDST near the boundaries is described by the directed linear forest model discussed in Section 3.

Initially we consider the ‘rooted’ case where we insert a point at the origin. Later we analyse the multiple sink (or ‘unrooted’) case, where we do not insert a point at the origin, in a similar way.

Fix $\sigma \in (1/2, 2/3)$. Let B_n denote the L-shaped boundary region $(0, 1]^2 \setminus (n^{-\sigma}, 1]^2$. Recall from (89) that $\mathcal{L}^\alpha(\mathcal{X}; R)$ denotes the contribution to the total weight of the MDST on \mathcal{X} from edges starting at points of $\mathcal{X} \cap R$. When \mathcal{X} is a random point set, set $\tilde{\mathcal{L}}^\alpha(\mathcal{X}; R) := \mathcal{L}^\alpha(\mathcal{X}; R) - E\mathcal{L}^\alpha(\mathcal{X}; R)$.

Theorem 7.1 *Suppose the partial order is \preceq^* . Then as $n \rightarrow \infty$ we have*

$$\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n^0; B_n) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha^{\{1\}} + \tilde{D}_\alpha^{\{2\}} \quad (\alpha \geq 1); \quad (101)$$

$$\tilde{\mathcal{L}}^\alpha(\mathcal{X}_n^0; B_n) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha^{\{1\}} + \tilde{D}_\alpha^{\{2\}} \quad (\alpha \geq 1), \quad (102)$$

where $\tilde{D}_\alpha^{\{1\}}, \tilde{D}_\alpha^{\{2\}}$ are independent random variables with the distribution of \tilde{D}_α given by the fixed-point equation (5) for $\alpha = 1$ and by (6) for $\alpha > 1$. Also, as

$n \rightarrow \infty$,

$$\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}} \quad (\alpha \geq 1); \quad (103)$$

$$\tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; B_n) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}} \quad (\alpha \geq 1), \quad (104)$$

where $\tilde{F}_\alpha^{\{1\}}, \tilde{F}_\alpha^{\{2\}}$ are independent random variables with the same distribution as \tilde{D}_1 for $\alpha = 1$ and with the distribution given by the fixed-point equation (7) for $\alpha > 1$. Also, as $n \rightarrow \infty$,

$$n^{(\alpha-1)/2} \mathcal{L}^\alpha(\mathcal{P}_n; B_n) \xrightarrow{L^1} 0 \quad (0 < \alpha < 1); \quad (105)$$

$$n^{(\alpha-1)/2} \mathcal{L}^\alpha(\mathcal{P}_n^0; B_n) \xrightarrow{L^1} 0 \quad (0 < \alpha < 1). \quad (106)$$

The idea behind the proof of Theorem 7.1 is to show that the MDSF near each of the two boundaries is close to a DLF system defined on a sequence of uniform random variables coupled to the points of the MDSF. To do this, we produce two explicit sequences of random variables on which we construct the DLF coupled to \mathcal{P}_n , a Poisson process of intensity n on $(0, 1]^2$, on which the MDSF is constructed.

Let B_n^x be the rectangle $(n^{-\sigma}, 1] \times (0, n^{-\sigma}]$, let B_n^y be the rectangle $(0, n^{-\sigma}] \times (n^{-\sigma}, 1]$, and let B_n^0 be the square $(0, n^{-\sigma}]^2$; see Figure 5. Then $B_n = B_n^0 \cup B_n^x \cup B_n^y$. Define the point processes

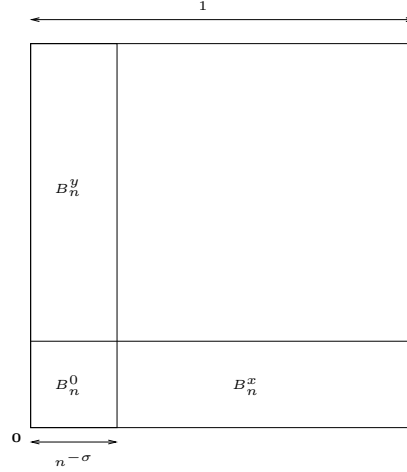


Figure 5: The boundary regions

$$\mathcal{V}_n^x := \mathcal{P}_n \cap (B_n^x \cup B_n^0), \quad \mathcal{V}_n^y := \mathcal{P}_n \cap (B_n^y \cup B_n^0), \quad \text{and} \quad \mathcal{V}_n^0 := \mathcal{P}_n \cap B_n^0. \quad (107)$$

Let $N_n^x := \text{card}(\mathcal{V}_n^x)$, $N_n^y := \text{card}(\mathcal{V}_n^y)$ and $N_n^0 := \text{card}(\mathcal{V}_n^0)$. List \mathcal{V}_n^x in order of increasing y -coordinate as $\mathbf{X}_i^x, i = 1, 2, \dots, N_n^x$. In coordinates, set $\mathbf{X}_i^x = (X_i^x, Y_i^x)$ for each i . Similarly, list \mathcal{V}_n^y in order of increasing x -coordinate as $\mathbf{X}_i^y = (X_i^y, Y_i^y)$, $i = 1, \dots, N_n^y$. Set $\mathcal{U}_n^x = (X_i^x, i = 1, 2, \dots, N_n^x)$ and $\mathcal{U}_n^y = (Y_i^y, i = 1, 2, \dots, N_n^y)$. Then \mathcal{U}_n^x and \mathcal{U}_n^y are sequences of uniform random variables in $(0, 1]$, on which we

may construct a DLF. Also, we write $\mathcal{U}_n^{x,0}$ for the sequence $(0, X_1^x, X_2^x, \dots, X_{N_n^x}^x)$, and $\mathcal{U}_n^{y,0}$ for the sequence $(0, Y_1^y, Y_2^y, \dots, Y_{N_n^y}^y)$.

With the total DLF/DLT weight functional $D^\alpha(\cdot)$ defined in Section 3 for random finite sequences in $(0, 1)$, the DLF weight $D^\alpha(\mathcal{U}_n^x)$ is coupled in a natural way to the MDSF contribution $\mathcal{L}^\alpha(\mathcal{V}_n^x)$, and likewise for $D^\alpha(\mathcal{U}_n^y)$ and $\mathcal{L}^\alpha(\mathcal{V}_n^y)$, for $D^\alpha(\mathcal{U}_n^{x,0})$ and $\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\})$, and for $D^\alpha(\mathcal{U}_n^{y,0})$ and $\mathcal{L}^\alpha(\mathcal{V}_n^y \cup \{\mathbf{0}\})$.

Lemma 7.1 *For any $\alpha \geq 1$, as $n \rightarrow \infty$,*

$$\mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x) \xrightarrow{L^2} 0, \text{ and } \mathcal{L}^\alpha(\mathcal{V}_n^y) - D^\alpha(\mathcal{U}_n^y) \xrightarrow{L^2} 0; \quad (108)$$

$$\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) - D^\alpha(\mathcal{U}_n^{x,0}) \xrightarrow{L^2} 0, \text{ and } \mathcal{L}^\alpha(\mathcal{V}_n^y \cup \{\mathbf{0}\}) - D^\alpha(\mathcal{U}_n^{y,0}) \xrightarrow{L^2} 0. \quad (109)$$

Further, for $0 < \alpha < 1$, as $n \rightarrow \infty$,

$$E \left[|\mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x)|^2 \right] = O(n^{2-2\sigma-2\alpha\sigma}), \quad (110)$$

and the corresponding result holds for \mathcal{V}_n^y and \mathcal{U}_n^y , and for the rooted cases (with the addition of the origin).

Proof. We approximate the MDSF in the region B_n by two DLFs, coupled to the MDSF. Consider \mathcal{V}_n^x ; the argument for \mathcal{V}_n^y is entirely analogous.

We have the set of points $\mathcal{V}_n^x = \{(X_i^x, Y_i^x), i = 1, \dots, N_n^x\}$. We construct the MDSF on these points, and construct the DLF on the x -coordinates, $\mathcal{U}_n^x = (X_i^x, i = 1, \dots, N_n^x)$. Consider any point (X_i^x, Y_i^x) . For any single point, either an edge exists from that point in both constructions, or in neither. Suppose an edge exists, that is suppose X_i^x is joined to a point $X_{D(i)}^x$, $D(i) < i$ in the DLF model, and (X_i^x, Y_i^x) to a point $(X_{N(i)}^x, Y_{N(i)}^x)$ in the MDST (we do not necessarily have $N(i) = D(i)$). By construction, we know that $|X_i^x - X_{D(i)}^x| \leq |X_i^x - X_{N(i)}^x|$, since $N(i) < i$ by the order of our points. It then follows that

$$\|(X_i^x, Y_i^x) - (X_{N(i)}^x, Y_{N(i)}^x)\|^\alpha \geq |X_i^x - X_{N(i)}^x|^\alpha \geq |X_i^x - X_{D(i)}^x|^\alpha,$$

and so we have established that, for all $\alpha > 0$,

$$D^\alpha(\mathcal{U}_n^x) \leq \mathcal{L}^\alpha(\mathcal{V}_n^x); \text{ and } D^\alpha(\mathcal{U}_n^{x,0}) \leq \mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}).$$

Now, by the construction of the MDST, we have that

$$\|(X_i^x, Y_i^x) - (X_{N(i)}^x, Y_{N(i)}^x)\| \leq \|(X_i^x, Y_i^x) - (X_{D(i)}^x, Y_{D(i)}^x)\|. \quad (111)$$

If $(x, y) \in (0, 1]^2$ then $\|(x, y)\| \leq x + y$, and by the Mean Value Theorem for the function $t \mapsto t^\alpha$, for $\alpha \geq 1$,

$$\|(x, y)\|^\alpha - x^\alpha \leq (x + y)^\alpha - x^\alpha \leq \alpha 2^{\alpha-1} y \quad (\alpha \geq 1).$$

Hence, for $\alpha \geq 1$,

$$\|(X_i^x, Y_i^x) - (X_{D(i)}^x, Y_{D(i)}^x)\|^\alpha - (X_i^x - X_{D(i)}^x)^\alpha \leq \alpha 2^{\alpha-1} (Y_i^x - Y_{D(i)}^x). \quad (112)$$

Then (111) and (112) yield, for $\alpha \geq 1$,

$$\|(X_i^x, Y_i^x) - (X_{N(i)}^x, Y_{N(i)}^x)\|^\alpha - (X_i^x - X_{D(i)}^x)^\alpha \leq \alpha 2^{\alpha-1} (Y_i^x - Y_{D(i)}^x).$$

Hence, for $\alpha \geq 1$,

$$0 \leq \mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x) \leq \alpha 2^{\alpha-1} \sum_{i=1}^{N_n^x} (Y_i^x - Y_{D(i)}^x).$$

Thus, for $\alpha \geq 1$,

$$\begin{aligned} 0 &\leq \mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x) \leq \alpha 2^{\alpha-1} N_n^x n^{-\sigma}; \\ \text{and } 0 &\leq \mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) - D^\alpha(\mathcal{U}_n^{x,0}) \leq \alpha 2^{\alpha-1} N_n^x n^{-\sigma}. \end{aligned} \quad (113)$$

We have $N_n^x \sim \text{Po}(n^{1-\sigma})$, so that since $\sigma > 1/2$, we have

$$E[(\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) - D^\alpha(\mathcal{U}_n^{x,0}))^2] \leq \alpha^2 2^{2\alpha-2} n^{-2\sigma} E[(N_n^x)^2] \rightarrow 0, \quad \alpha \geq 1.$$

An entirely analogous argument leads to the same statement for \mathcal{U}_n^y and \mathcal{V}_n^y , and we obtain (108), and (109) in identical fashion.

We now consider $0 < \alpha < 1$. By the concavity of the function $t \mapsto t^\alpha$ for $\alpha < 1$, we have for $x > 0, y > 0$ that

$$\|(x, y)\|^\alpha - x^\alpha \leq (x + y)^\alpha - x^\alpha \leq y^\alpha \quad (0 < \alpha < 1).$$

Then, by a similar argument to (113) in the $\alpha \geq 1$ case, we obtain

$$0 \leq \mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x) \leq N_n^x n^{-\alpha\sigma}.$$

Then (110) follows since $N_n^x \sim \text{Po}(n^{1-\sigma})$, and the rooted case is similar. \square

Lemma 7.2 *Suppose \tilde{D}_1 has distribution given by (5), \tilde{D}_α , $\alpha > 1$, has distribution given by (6), and \tilde{F}_α , $\alpha > 1$, has distribution given by (7). Then as $n \rightarrow \infty$,*

$$\tilde{\mathcal{L}}^1(\mathcal{V}_n^x \cup \{\mathbf{0}\}) \xrightarrow{\mathcal{D}} \tilde{D}_1, \quad \text{and} \quad \tilde{\mathcal{L}}^1(\mathcal{V}_n^x) \xrightarrow{\mathcal{D}} \tilde{D}_1; \quad (114)$$

$$\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha, \quad \text{and} \quad \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha \quad (\alpha > 1). \quad (115)$$

Moreover, (114) and (115) also hold with \mathcal{V}_n^x replaced by \mathcal{V}_n^y .

Proof. As usual we present the argument for \mathcal{V}_n^x only, since the result for \mathcal{V}_n^y follows in the same manner. First consider the $\alpha > 1$ case. We have the distributional equality

$$\mathcal{L}(D^\alpha(\mathcal{U}_n^{x,0}) | N_n^x = m) = \mathcal{L}(D^\alpha(\mathcal{U}_m^0)); \quad \mathcal{L}(D^\alpha(\mathcal{U}_n^x) | N_n^x = m) = \mathcal{L}(D^\alpha(\mathcal{U}_m)).$$

But N_n^x is Poisson with mean $n^{1-\sigma}$, and so tends to infinity almost surely. Thus by Theorem 3.1 (ii), $D^\alpha(\mathcal{U}_n^{x,0}) \xrightarrow{\mathcal{D}} D_\alpha$ and $D^\alpha(\mathcal{U}_n^x) \xrightarrow{\mathcal{D}} F_\alpha$ as $n \rightarrow \infty$, and so by Lemma 7.1 and Slutsky's theorem, we obtain

$$\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) \xrightarrow{\mathcal{D}} D_\alpha \quad \text{and} \quad \mathcal{L}^\alpha(\mathcal{V}_n^x) \xrightarrow{\mathcal{D}} F_\alpha \quad \text{as } n \rightarrow \infty. \quad (116)$$

Also, $E[D^\alpha(\mathcal{U}_n^{x,0})] \rightarrow (\alpha - 1)^{-1}$ by (24), so by Lemma 7.1 and Proposition 3.3, $E[\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\})] \rightarrow (\alpha - 1)^{-1} = E[D_\alpha]$. Similarly, by (33), Lemma 7.1 and Proposition 3.4, $E[\mathcal{L}^\alpha(\mathcal{V}_n^x)] \rightarrow (\alpha(\alpha - 1))^{-1} = E[F_\alpha]$. Hence, (116) still holds with the centred variables, i.e., (115) holds.

Now suppose $\alpha = 1$. Since N_n^x is Poisson with parameter $n^{1-\sigma}$, Lemma 3.7 (i), with $t = n^{1-\sigma}$, then shows that $\tilde{D}^1(\mathcal{U}_n^{x,0}) \xrightarrow{\mathcal{D}} \tilde{D}_1$ as $n \rightarrow \infty$. Slutsky's theorem with Lemma 7.1 then implies that $\tilde{\mathcal{L}}^1(\mathcal{V}_n^x \cup \{\mathbf{0}\}) \xrightarrow{\mathcal{D}} \tilde{D}_1$. In the same way we obtain $\tilde{\mathcal{L}}^1(\mathcal{V}_n^x) \xrightarrow{\mathcal{D}} \tilde{D}_1$, this time using part (ii) instead of part (i) of Lemma 3.7, along with Proposition 3.7. \square

Note that $D^\alpha(\mathcal{U}_n^x)$ and $D^\alpha(\mathcal{U}_n^y)$ are not independent. To deal with this, we define

$$\tilde{\mathcal{V}}_n^x := \mathcal{P}_n \cap B_n^x, \quad \text{and} \quad \tilde{\mathcal{V}}_n^y := \mathcal{P}_n \cap B_n^y.$$

Also, recall the definition of \mathcal{V}_n^0 at (107). Let $\tilde{N}_n^x := \text{card}(\tilde{\mathcal{V}}_n^x)$ and $\tilde{N}_n^y := \text{card}(\tilde{\mathcal{V}}_n^y)$. Since B_n^x and B_n^y are disjoint, $\mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^x)$ and $\mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^y)$ are independent, by the spatial independence property of the Poisson process \mathcal{P}_n .

Lemma 7.3 *Suppose $\alpha > 0$. Then:*

(i) *As $n \rightarrow \infty$,*

$$\mathcal{L}^\alpha(\mathcal{V}_n^x) - \mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^x) \xrightarrow{L^1} 0, \quad \text{and} \quad \mathcal{L}^\alpha(\mathcal{V}_n^y) - \mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^y) \xrightarrow{L^1} 0; \quad (117)$$

$$\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) - \mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^x \cup \{\mathbf{0}\}) \xrightarrow{L^1} 0, \quad \text{and} \quad \mathcal{L}^\alpha(\mathcal{V}_n^y \cup \{\mathbf{0}\}) - \mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^y \cup \{\mathbf{0}\}) \xrightarrow{L^1} 0. \quad (118)$$

(ii) *As $n \rightarrow \infty$, we have $\mathcal{L}^\alpha(\mathcal{V}_n^0) \xrightarrow{L^1} 0$, and $\mathcal{L}^\alpha(\mathcal{V}_n^0 \cup \{\mathbf{0}\}) \xrightarrow{L^1} 0$.*

Proof. We first prove (i). We give only the argument for \mathcal{V}_n^x ; that for \mathcal{V}_n^y is analogous. Set $\Delta := \mathcal{L}^\alpha(\mathcal{V}_n^x) - \mathcal{L}^\alpha(\tilde{\mathcal{V}}_n^x)$. Let $\beta = (\sigma + (1/2))/2$. Then $1/2 < \beta < \sigma$.

Assume without loss of generality that \mathcal{P}_n is the restriction to $(0, 1]^2$ of a homogeneous Poisson process \mathcal{H}_n of intensity n on \mathbf{R}^2 . Let $\mathbf{X}^- = (X^-, Y^-)$ be the point of $\mathcal{H}_n \cap ((0, n^{-\beta}] \times (0, \infty))$ with minimal y -coordinate. Then X^- is uniform on $(0, n^{-\beta}]$. Let E_n be the event that $X^- > 3n^{-\sigma}$; then $P[E_n^c] = 3n^{\beta-\sigma}$ for n large enough.

Let Δ_1 be the contribution to Δ from edges starting at points in $(0, n^{-\beta}] \times (0, n^{-\sigma}]$. Then the absolute value of Δ_1 is bounded by the product of $(\sqrt{2}n^{-\beta})^\alpha$ and the number of points of \mathcal{P}_n in $(0, n^{-\beta}] \times (0, n^{-\sigma}]$. Hence, for any $\alpha > 0$,

$$\begin{aligned} E[|\Delta_1|] &\leq (\sqrt{2}n^{-\beta})^\alpha E\left[\text{card}\left(\mathcal{P}_n \cap ((0, n^{-\beta}] \times (0, n^{-\sigma}])\right)\right] \\ &= 2^{\alpha/2} n^{1-\beta-\sigma-\alpha\beta} \rightarrow 0. \end{aligned} \quad (119)$$

Let $\Delta_2 := \Delta - \Delta_1$, the contribution to Δ from edges starting at points in $(n^{-\beta}, 1] \times (0, n^{-\sigma}]$. Then by the triangle inequality, if E_n occurs then these edges are unaffected by points in B_n^0 , so that Δ_2 is zero if E_n occurs. Also, only minimal elements of $\mathcal{P}_n \cap (n^{-\beta}, 1] \times (0, n^{-\sigma}]$ can possibly have their directed nearest neighbour in $(0, n^{-\sigma}] \times (0, n^{-\sigma}]$; hence, if M_n denotes the number of such minimal elements then $|\Delta_2|$ is bounded by $2^{\alpha/2} M_n$. Hence, using (86), we obtain

$$E[|\Delta_2|] \leq 2^{\alpha/2} P[E_n^c] E[M_n] = O(n^{\beta-\sigma} \log n)$$

which tends to zero. Combined with (119), this gives us (117). The same argument gives us (118).

For (ii), note that

$$E[\mathcal{L}^\alpha(\mathcal{V}_n^0)] \leq (\sqrt{2}n^{-\sigma})^\alpha E[N_n^0] = 2^{\alpha/2} n^{1-2\sigma-\sigma\alpha} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

for any $\alpha > 0$. Thus $\mathcal{L}^\alpha(\mathcal{V}_n^0) \xrightarrow{L^1} 0$, and similarly $\mathcal{L}^\alpha(\mathcal{V}_n^0 \cup \{\mathbf{0}\}) \xrightarrow{L^1} 0$. \square

In proving our next lemma (and again later on) we use the following elementary fact. If $N(n)$ is Poisson with parameter n , then as $n \rightarrow \infty$ we have

$$E[|N(n) - n| \log \max(N(n), n)] = O(n^{1/2} \log n). \quad (120)$$

To see this, set $Y_n := |N(n) - n| \log \max(N(n), n)$. Then $Y_n \mathbf{1}_{\{N(n) \leq 2n\}} \leq |N(n) - n| \log(2n)$, and the expectation of this is $O(n^{1/2} \log n)$ by Jensen's inequality since $\text{Var}(N(n)) = n$. On the other hand, the Cauchy-Schwarz inequality shows that $E[Y_n \mathbf{1}_{\{N(n) > 2n\}}] \rightarrow 0$, and (120) follows.

We now state a lemma for coupling \mathcal{X}_n and \mathcal{P}_n . The $\alpha \geq 1$ part will be used in the proof of Theorem 7.1. The $0 < \alpha < 1$ part will be needed later, in the proof of Theorem 2.2. As in Section 6, let $S_{0,n}$ denote the ‘inner’ region $(n^{\varepsilon-1/2}, 1]^2$, with $\varepsilon \in (0, 1/2)$ a constant. The boundary region B_n is disjoint from $S_{0,n}$; let C_n denote the intermediate region $(0, 1]^2 \setminus (B_n \cup S_{0,n})$, so that $B_n \cup C_n = (0, 1]^2 \setminus S_{0,n}$.

Lemma 7.4 *There exists a coupling of \mathcal{X}_n and \mathcal{P}_n such that:*

(i) *For $0 < \alpha < 1$, provided $\varepsilon < (1 - \alpha)/2$, we have that as $n \rightarrow \infty$,*

$$n^{(\alpha-1)/2} E[|\mathcal{L}^\alpha(\mathcal{X}_n; B_n \cup C_n) - \mathcal{L}^\alpha(\mathcal{P}_n; B_n \cup C_n)|] \rightarrow 0 \quad (121)$$

and

$$n^{(\alpha-1)/2} E[|\mathcal{L}^\alpha(\mathcal{X}_n^0; B_n \cup C_n) - \mathcal{L}^\alpha(\mathcal{P}_n^0; B_n \cup C_n)|] \rightarrow 0. \quad (122)$$

(ii) *For $\alpha \geq 1$, we have that as $n \rightarrow \infty$,*

$$E[|\mathcal{L}^\alpha(\mathcal{X}_n; B_n) - \mathcal{L}^\alpha(\mathcal{P}_n; B_n)|] \rightarrow 0 \quad (123)$$

and

$$E[|\mathcal{L}^\alpha(\mathcal{X}_n^0; B_n) - \mathcal{L}^\alpha(\mathcal{P}_n^0; B_n)|] \rightarrow 0. \quad (124)$$

Proof. We couple \mathcal{X}_n and \mathcal{P}_n in the following standard way. Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots$ be independent uniform random vectors on $(0, 1]^2$, and let $N(n) \sim \text{Po}(n)$ be independent of $(\mathbf{X}_1, \mathbf{X}_2, \dots)$. For $m \in \mathbf{N}$ (and in particular for $m = n$) set $\mathcal{X}_m := \{\mathbf{X}_1, \dots, \mathbf{X}_m\}$; set $\mathcal{P}_n := \{\mathbf{X}_1, \dots, \mathbf{X}_{N(n)}\}$.

For each $m \in \mathbf{N}$, let Y_m denote the in-degree of vertex \mathbf{X}_m in the MDST on \mathcal{X}_m . Suppose $\mathbf{X}_m = \mathbf{x}$. Then an upper bound for Y_m is provided by the number of minimal elements of the restriction of \mathcal{X}_{m-1} to the rectangle $\{\mathbf{y} \in (0, 1]^2 : \mathbf{x} \preccurlyeq^* \mathbf{y}\}$. Hence, conditional on $\mathbf{X}_m = \mathbf{x}$ and on there being k points of \mathcal{X}_{m-1} in this rectangle, the expected value of Y_m is bounded by the expected number of minimal elements in a random uniform sample of k points in this rectangle, and hence (see (86)) by $1 + \log k$. Hence, given the value of \mathbf{X}_m , the conditional expectation of Y_m is bounded by $1 + \log m$.

First we prove the statements in part (i) ($0 < \alpha < 1$). Suppose $\varepsilon < (1 - \alpha)/2$. Then

$$|\mathcal{L}^\alpha(\mathcal{X}_m; B_n \cup C_n) - \mathcal{L}^\alpha(\mathcal{X}_{m-1}; B_n \cup C_n)| \leq 2^{\alpha/2}(Y_m + 1)\mathbf{1}\{\mathbf{X}_m \in B_n \cup C_n\}. \quad (125)$$

Since $B_n \cup C_n$ has area $2n^{\varepsilon-1/2} - n^{2\varepsilon-1}$, we obtain

$$E[(Y_m + 1)\mathbf{1}\{\mathbf{X}_m \in B_n \cup C_n\}] \leq (2 + \log m)2n^{\varepsilon-1/2}.$$

Hence, by (125) there is a constant C such that

$$\begin{aligned} n^{(\alpha-1)/2} E[|(\mathcal{L}^\alpha(\mathcal{P}_n; B_n \cup C_n) - \mathcal{L}^\alpha(\mathcal{X}_n; B_n \cup C_n))||N(n)] \\ \leq C|N(n) - n| \log(\max(N(n), n))n^{(\alpha+2\varepsilon-2)/2}, \end{aligned}$$

and since we assume $\alpha + 2\varepsilon < 1$, by (120) the expected value of the right hand side tends to zero as $n \rightarrow \infty$, and we obtain (121). Likewise in the rooted case (122).

Now we prove part (ii). For $\alpha \geq 1$, we have

$$|\mathcal{L}^\alpha(\mathcal{X}_m; B_n) - \mathcal{L}^\alpha(\mathcal{X}_{m-1}; B_n)| \leq 2^{\alpha/2}(Y_m + 1)\mathbf{1}\{\mathbf{X}_m \in B_n\}. \quad (126)$$

Since B_n has area $2n^{-\sigma} - n^{-2\sigma}$, by (126) there is a constant C such that

$$E[|(\mathcal{L}^\alpha(\mathcal{P}_n; B_n) - \mathcal{L}^\alpha(\mathcal{X}_n; B_n))||N(n)] \leq C|N(n) - n| \log(\max(N(n), n))n^{-\sigma},$$

and since $\sigma > 1/2$, by (120) the expected value of the right hand side tends to zero as $n \rightarrow \infty$, and we obtain (123). We get (124) similarly. \square

Proof of Theorem 7.1. Suppose $\alpha \geq 1$. We have that

$$\tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^x) = \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) + (\tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^x) - \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x)).$$

The final bracket converges to zero in probability, by Lemma 7.3 (i). Thus by Lemma 7.2 and Slutsky's theorem, we obtain $\tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^x) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha$ (where we have $\tilde{F}_1 \stackrel{\mathcal{D}}{=} \tilde{D}_1$). Now

$$\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) + \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y) = \tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^x) + \tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^y) + (\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) - \tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^x)) + (\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y) - \tilde{\mathcal{L}}^\alpha(\tilde{\mathcal{V}}_n^y)).$$

The last two brackets converge to zero in probability, by Lemma 7.3 (i). Then the independence of $\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x)$ and $\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y)$ and another application of Slutsky's theorem yield

$$\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) + \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y) \xrightarrow{\mathcal{D}} \tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}},$$

where $\tilde{F}_\alpha^{\{1\}}$ and $\tilde{F}_\alpha^{\{2\}}$ are independent copies of \tilde{F}_α . Similarly,

$$\tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) + \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y \cup \{\mathbf{0}\}) \xrightarrow{\mathcal{D}} \tilde{D}_\alpha^{\{1\}} + \tilde{D}_\alpha^{\{2\}}.$$

Finally, since $\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n) = \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^x) + \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^y) - \tilde{\mathcal{L}}^\alpha(\mathcal{V}_n^0)$ (with a similar statement including the origin) Lemma 7.3 (ii) and Slutsky's theorem complete the proof of (101) and (103).

To deduce (102) and (104), assume without loss of generality that \mathcal{X}_n and \mathcal{P}_n are coupled in the manner of Lemma 7.4. Then $\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n) - \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; B_n)$ tends to zero in probability by (123), and $\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n^0; B_n) - \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n^0; B_n)$ tends to zero in probability by (124). Hence by Slutsky's theorem, the convergence results (101) and (103) carry through to the binomial point process case, i.e., (102) and (104) hold.

Now suppose $0 < \alpha < 1$. Then (110) gives us

$$E \left[\left| n^{(\alpha-1)/2} (\mathcal{L}^\alpha(\mathcal{V}_n^x) - D^\alpha(\mathcal{U}_n^x)) \right|^2 \right] = O \left(n^{(\alpha+1)(1-2\sigma)} \right), \quad (127)$$

which tends to 0 as $n \rightarrow \infty$, since $\sigma > 1/2$. Likewise for the rooted case,

$$E \left[\left| n^{(\alpha-1)/2} (\mathcal{L}^\alpha(\mathcal{V}_n^x \cup \{\mathbf{0}\}) - D^\alpha(\mathcal{U}_n^{x,0})) \right|^2 \right] = O \left(n^{(\alpha+1)(1-2\sigma)} \right), \quad (128)$$

By Proposition 3.2 we have

$$E[n^{(\alpha-1)/2} D^\alpha(\mathcal{U}_n^x)] = O(n^{(\alpha-1)/2} E[(N_n^x)^{1-\alpha}]) = O(n^{(\alpha-1)(\sigma-1/2)}) \rightarrow 0,$$

and combined with (127) this completes the proof of (105). Similarly, by Proposition 3.1,

$$E[n^{(\alpha-1)/2} D^\alpha(\mathcal{U}_n^{x,0})] = O(n^{(\alpha-1)/2} E[(N_n^x)^{1-\alpha}]) = O(n^{(\alpha-1)(\sigma-1/2)}) \rightarrow 0,$$

and combined with (128) this gives us (106). \square

8 Proof of Theorem 2.2

Let $\sigma \in (1/2, 2/3)$. Let $\varepsilon > 0$ with

$$\varepsilon < \min(1/2, (1-\sigma)/3, (3-4\sigma)/10, (2-3\sigma)/8). \quad (129)$$

In addition, if $0 < \alpha < 1$, we impose the further condition that $\varepsilon < (1-\alpha)/2$. As in Section 6, denote by $S_{0,n}$ the region $(n^{\varepsilon-1/2}, 1]^2$. As in Section 7, let B_n denote the region $(0, 1]^2 \setminus (n^{-\sigma}, 1]^2$, and let C_n denote $(0, 1]^2 \setminus (B_n \cup S_{0,n})$.

We know from Sections 6 and 7 that, for large n , the weight of edges starting in $S_{0,n}$ satisfies a central limit theorem, and the weight of edges starting in B_n can be approximated by the directed linear forest. We shall show in Lemmas 8.2 and 8.3 that (with a suitable scaling factor for $\alpha < 1$) the contribution to the total weight from points in C_n has variance converging to zero. To complete the proof of Theorem 2.2 in the Poisson case, we shall show that the lengths from B_n and $S_{0,n}$ are asymptotically independent by virtue of the fact that the configuration of points in C_n is (with probability approaching one) sufficient to ensure that the configuration of points in B_n has no effect on the edges from points in $S_{0,n}$. To extend the result to the binomial point process case, we shall use a de-Poissonization argument related to that used in [17].

First consider the region C_n . We naturally divide this into three regions. Let

$$C_n^x := (n^{\varepsilon-1/2}, 1] \times (n^{-\sigma}, n^{\varepsilon-1/2}], \quad C_n^y := (n^{-\sigma}, n^{\varepsilon-1/2}] \times (n^{\varepsilon-1/2}, 1], \\ C_n^0 := (n^{-\sigma}, n^{\varepsilon-1/2}]^2.$$

Also, as in Section 7, let

$$B_n^x := (n^{-\sigma}, 1] \times (0, n^{-\sigma}], \quad B_n^y := (0, n^{-\sigma}] \times (n^{-\sigma}, 1], \quad B_n^0 := (0, n^{-\sigma}]^2.$$

We divide the C_n and B_n into rectangular cells as follows (see Figure 6.) We leave C_n^0 undivided. We set

$$k_n := \lfloor n^{1-\sigma-2\varepsilon} \rfloor \tag{130}$$

and divide C_n^x lengthways into k_n cells. For each cell,

$$\text{width} = (1 - n^{\varepsilon-1/2})/k_n \sim n^{2\varepsilon+\sigma-1}; \quad \text{height} = n^{\varepsilon-1/2} - n^{-\sigma} \sim n^{\varepsilon-1/2}. \tag{131}$$

Label these cells Γ_i^x for $i = 1, 2, \dots, k_n$ from left to right. For each cell Γ_i^x , define the adjoining cell of B_n^x , formed by extending the vertical edges of Γ_i^x , to be β_i^x . The cells β_i^x then have width $(1 - n^{\varepsilon-1/2})/k_n \sim n^{2\varepsilon+\sigma-1}$ and height $n^{-\sigma}$.

In a similar way we divide C_n^y into k_n cells Γ_i^y of height $(1 - n^{\varepsilon-1/2})/k_n$ and width $n^{\varepsilon-1/2} - n^{-\sigma}$, and divide B_n^y into the corresponding cells β_i^y , $i = 1, \dots, k_n$.

For $i = 2, \dots, k_n$, let $E_{x,i}$ denote the event that the cell β_{i-1}^x contains at least one point of \mathcal{P}_n , and let $E_{y,i}$ denote the event that β_{i-1}^y contains at least one point of \mathcal{P}_n .

Lemma 8.1 *For n sufficiently large, and for $1 \leq j < i \leq k_n$ with $i - j > 3$, if $E_{x,i}$ (respectively $E_{y,i}$) occurs then no point in the cell Γ_i^x (respectively Γ_i^y) has a directed nearest neighbour in the cell Γ_j^x or β_j^x (Γ_j^y or β_j^y).*

Proof. Consider a point X , say, in cell Γ_i^x in C_n^x . Given $E_{x,i}$, we know that there is a point, Y say, in the cell β_{i-1}^x to the left of the β_i^x cell immediately below Γ_i^x , such that $Y \preceq^* X$, but the difference in x -coordinates between X and Y is no more than twice the width of a cell. So, by the triangle inequality, we have

$$\|X - Y\| \leq 2(1 - n^{\varepsilon-1/2})/k_n + n^{\varepsilon-1/2} \sim 2n^{2\varepsilon+\sigma-1}, \tag{132}$$

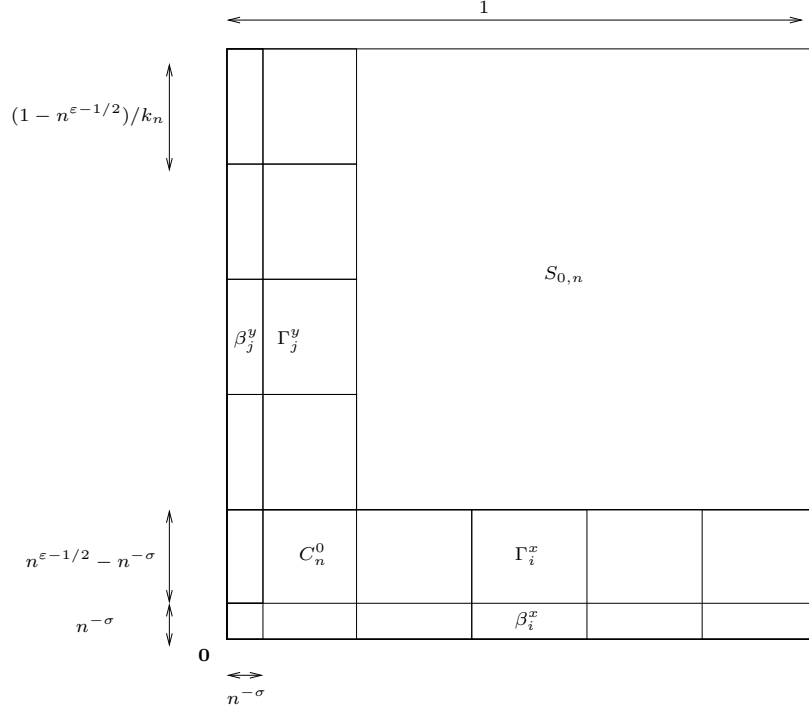


Figure 6: The regions of $[0, 1]^2$.

since $\sigma > 1/2$. Now, consider a point Z in a cell Γ_j^x or β_j^x with $j \leq i - 4$. In this case, the difference in x -coordinates between X and Z is at least the width of 3 cells, so that

$$\|X - Z\| \geq 3(1 - n^{\varepsilon-1/2})/k_n \sim 3n^{2\varepsilon+\sigma-1}. \quad (133)$$

Comparing (132) and (133), we see that X is not connected to Z , which completes the proof. \square

Recall from (89) that for a point set $\mathcal{S} \subset \mathbf{R}^2$ and a region $R \subseteq \mathbf{R}^2$, $\mathcal{L}^\alpha(\mathcal{S}; R)$ denotes the total weight of edges of the MDSF on \mathcal{S} which originate in the region R .

Lemma 8.2 *As $n \rightarrow \infty$, we have that*

$$\text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n; C_n)] \rightarrow 0 \quad \text{and} \quad \text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n^0; C_n)] \rightarrow 0 \quad (\alpha \geq 1); \quad (134)$$

$$\text{Var}[n^{(\alpha-1)/2} \mathcal{L}^\alpha(\mathcal{P}_n; C_n)] \rightarrow 0 \quad (0 < \alpha < 1); \quad (135)$$

$$\text{Var}[n^{(\alpha-1)/2} \mathcal{L}^\alpha(\mathcal{P}_n^0; C_n)] \rightarrow 0 \quad (0 < \alpha < 1). \quad (136)$$

Proof. For ease of notation, write $X_i = \mathcal{L}^\alpha(\mathcal{P}_n; \Gamma_i^x)$ and $Y_i = \mathcal{L}^\alpha(\mathcal{P}_n; \Gamma_i^y)$, for $i = 1, 2, \dots, k_n$. Also let $Z = \mathcal{L}^\alpha(\mathcal{P}_n; C_n^0)$. Then

$$\text{Var}[\mathcal{L}^\alpha(\mathcal{P}_n; C_n)] = \text{Var}\left[Z + \sum_{i=1}^{k_n} X_i + \sum_{i=1}^{k_n} Y_i\right]. \quad (137)$$

Let N_i^x , N_i^y , N_0 , respectively, denote the number of points of \mathcal{P}_n in Γ_i^x , Γ_i^y , C_n^0 , respectively. Then by (131), N_i^x is Poisson with parameter asymptotic to $n^{3\varepsilon+\sigma-1/2}$, while $N_1^x + N_1^y + N_0$ is Poisson with parameter asymptotic to $2n^{3\varepsilon+\sigma-1/2}$; hence as $n \rightarrow \infty$ and we have

$$E[(N_i^x)^2] \sim n^{6\varepsilon+2\sigma-1}, \quad E[(N_1^x + N_1^y + N_0)^2] \sim 4n^{6\varepsilon+2\sigma-1}. \quad (138)$$

Edges from points in $\Gamma_1^x \cap \Gamma_1^y \cap C_n^0$ are of length at most $2n^{2\varepsilon+\sigma-1}$, and hence,

$$\begin{aligned} \text{Var}[X_1 + Y_1 + Z] &\leq (2n^{2\varepsilon+\sigma-1})^{2\alpha} E[(N_1^x + N_1^y + N_0)^2] \\ &\sim 2^{2+2\alpha} n^{6\varepsilon+2\sigma-1+2\alpha(2\varepsilon+\sigma-1)}. \end{aligned} \quad (139)$$

For $\alpha \geq 1$, since ε is small (129), the expression (139) is $O(n^{10\varepsilon+4\sigma-3})$ and in fact tends to zero, so that

$$\text{Var}(X_1 + Y_1 + Z) \rightarrow 0 \quad (\alpha \geq 1). \quad (140)$$

By Lemma 8.1 and (132), given $E_{x,i}$, an edge from a point of Γ_i^x can be of length no more than $3n^{2\varepsilon+\sigma-1}$. Thus using (138) we have

$$\begin{aligned} \text{Var}[X_i \mathbf{1}\{E_{x,i}\}] &\leq E[X_i^2 \mathbf{1}\{E_{x,i}\}] \leq (3n^{2\varepsilon+\sigma-1})^{2\alpha} E[(N_i^x)^2] \\ &= O(n^{6\varepsilon+2\sigma-1+2\alpha(2\varepsilon+\sigma-1)}). \end{aligned} \quad (141)$$

Next, observe that $\text{Cov}[X_i \mathbf{1}\{E_{x,i}\}, X_j \mathbf{1}\{E_{x,j}\}] = 0$ for $i - j > 3$, since by Lemma 8.1, $X_i \mathbf{1}\{E_{x,i}\}$ is determined by the restriction of \mathcal{P}_n to the union of the regions $\Gamma_\ell^x \cup \beta_\ell^x$, $i - 3 \leq \ell \leq i$. Thus by (130), Cauchy-Schwarz and (141), we obtain

$$\begin{aligned} \text{Var}\left[\sum_{i=2}^{k_n} X_i \mathbf{1}\{E_{x,i}\}\right] &= \sum_{i=2}^{k_n} \text{Var}[X_i \mathbf{1}\{E_{x,i}\}] \\ &\quad + \sum_{i=2}^{k_n} \sum_{j: 1 \leq |j-i| \leq 3} \text{Cov}[X_i \mathbf{1}\{E_{x,i}\}, X_j \mathbf{1}\{E_{x,j}\}] \\ &= O(n^{4\varepsilon+\sigma+2\alpha(2\varepsilon+\sigma-1)}). \end{aligned} \quad (142)$$

For $\alpha \geq 1$, the bound in (142) tends to zero as $n \rightarrow \infty$, since $1/2 < \sigma < 2/3$ and ε is small (129).

By (130), the cells β_i^x , $i = 1, \dots, k_n$, have width asymptotic to $n^{2\varepsilon+\sigma-1}$ and height $n^{-\sigma}$, so the mean number of points of \mathcal{P}_n in one of these cells is asymptotic to $n^{2\varepsilon}$; hence for any cell β_i^x or β_i^y , $i = 1, \dots, k_n$, the probability that the cell

contains no point of \mathcal{P}_n is given by $\exp\{-n^{2\varepsilon}(1+o(1))\}$. Hence for n large enough, and $i = 2, \dots, k_n$, we have $P[E_{x,i}^c] \leq \exp(-n^\varepsilon)$, and hence by (138),

$$\begin{aligned} \text{Var}[X_i \mathbf{1}\{E_{x,i}^c\}] &\leq E[X_i^2 | E_{x,i}^c] P[E_{x,i}^c] \leq 2^\alpha E[(N_i^x)^2] P[E_{x,i}^c] \\ &= O(n^{6\varepsilon+2\sigma-1} \exp(-n^\varepsilon)). \end{aligned} \quad (143)$$

Hence by Cauchy-Schwarz we have

$$\begin{aligned} \text{Var} \left[\sum_{i=2}^{k_n} X_i \mathbf{1}\{E_{x,i}^c\} \right] &= \sum_{i=2}^{k_n} \text{Var}[X_i \mathbf{1}\{E_{x,i}^c\}] + \sum_{i \neq j} \text{Cov}[X_i \mathbf{1}\{E_{x,i}^c\}, X_j \mathbf{1}\{E_{x,j}^c\}] \\ &= O(k_n^2 n^{6\varepsilon+2\sigma-1} \exp(-n^\varepsilon)) \rightarrow 0, \end{aligned} \quad (144)$$

as $n \rightarrow \infty$. Then by (142), (144), and the analogous estimates for Y_i , along with the Cauchy-Schwarz inequality, we obtain for $\alpha \geq 1$ that

$$\text{Var} \left[\sum_{i=2}^{k_n} X_i \mathbf{1}\{E_{x,i}\} + \sum_{i=2}^{k_n} Y_i \mathbf{1}\{E_{y,i}\} + \sum_{i=2}^{k_n} X_i \mathbf{1}\{E_{x,i}^c\} + \sum_{i=2}^{k_n} Y_i \mathbf{1}\{E_{y,i}^c\} \right] \rightarrow 0, \quad (145)$$

as $n \rightarrow \infty$. By (137) with (140), (145), and Cauchy-Schwarz again, we obtain the first part of (134). The argument for \mathcal{P}_n^0 is the same as for \mathcal{P}_n , so we have (134).

Now suppose $0 < \alpha < 1$. We obtain (135) and (136) in a similar way to (134), since (139) implies that

$$\text{Var}(n^{(\alpha-1)/2}(X_1 + Y_1 + Z)) = O(n^{6\varepsilon+2\sigma-2+\alpha(4\varepsilon+2\sigma-1)})$$

and (142) implies

$$\text{Var} \left(n^{(\alpha-1)/2} \sum_{i=2}^{k_n} X_i \mathbf{1}\{E_{x,i}\} \right) = O(n^{4\varepsilon+\sigma-1+\alpha(4\varepsilon+2\sigma-1)}),$$

and both of these bounds tend to zero when $0 < \alpha < 1$, $1/2 < \sigma < 2/3$, and ε is small (129). \square

To prove those parts of Theorem 2.2 which refer to the binomial process \mathcal{X}_n , we need further results comparing the processes \mathcal{X}_n and \mathcal{P}_n when they are coupled as in Lemma 7.4.

Lemma 8.3 *Suppose $\alpha \geq 1$. With \mathcal{X}_n and \mathcal{P}_n coupled as in Lemma 7.4, we have that as $n \rightarrow \infty$*

$$\mathcal{L}^\alpha(\mathcal{X}_n; C_n) - \mathcal{L}^\alpha(\mathcal{P}_n; C_n) \xrightarrow{L^1} 0 \quad \text{and} \quad \mathcal{L}^\alpha(\mathcal{X}_n^0; C_n) - \mathcal{L}^\alpha(\mathcal{P}_n^0; C_n) \xrightarrow{L^1} 0. \quad (146)$$

Proof. Let \mathcal{P}_n and \mathcal{X}_m ($m \in \mathbb{N}$) be coupled as described in Lemma 7.4. Given n , for $m \in \mathbb{N}$ define the event

$$E_{m,n} := \cap_{1 \leq i \leq k_n} (\{\mathcal{X}_{m-1} \cap \beta_i^x \neq \emptyset\} \cap \{\mathcal{X}_{m-1} \cap \beta_i^y \neq \emptyset\}),$$

with the sub-cells β_i^x and β_i^y of B_n as defined near the start of Section 8. Then by similar arguments to those for $P[E_{x,i}^c]$ above, we have

$$P[E_{m,n}^c] = O(n^{1-\sigma-2\varepsilon} \exp(-n^\varepsilon/2)), \quad m \geq n/2 + 1.$$

As in the proof of Lemma 7.4, let Y_m denote the in-degree of vertex \mathbf{X}_m in the MDST on \mathcal{X}_m . Then

$$|\mathcal{L}^\alpha(\mathcal{X}_m; C_n) - \mathcal{L}^\alpha(\mathcal{X}_{m-1}; C_n)| \leq (Y_m + 1) \mathbf{1}\{\mathbf{X}_m \in C_n\} \left((3n^{2\varepsilon+\sigma-1})^\alpha + 2^{\alpha/2} \mathbf{1}\{E_{m,n}^c\} \right).$$

Thus, given $N(n)$,

$$\begin{aligned} |\mathcal{L}^\alpha(\mathcal{X}_n; C_n) - \mathcal{L}^\alpha(\mathcal{P}_n; C_n)| &\leq \sum_{m=\min(N(n), n)}^{\max(N(n), n)} (Y_m + 1) \mathbf{1}\{\mathbf{X}_m \in C_n\} \\ &\quad \times \left(3^\alpha n^{\alpha(2\varepsilon+\sigma-1)} + 2^{\alpha/2} \mathbf{1}\{E_{m,n}^c\} \right). \end{aligned}$$

Since C_n has area less than $2n^{\varepsilon-1/2}$, by (86) there exists a constant C such that, for n sufficiently large and $N(n) \geq n/2 + 1$,

$$\begin{aligned} E[|(\mathcal{L}^\alpha(\mathcal{X}_n; C_n) - \mathcal{L}^\alpha(\mathcal{P}_n; C_n))| | N(n)] &\leq 2^{\alpha/2} n \mathbf{1}_{\{N(n) < n/2+1\}} \\ &+ C |N(n) - n| \log(\max(N(n), n)) n^{\alpha(2\varepsilon+\sigma-1)+\varepsilon-1/2} \mathbf{1}_{\{N(n) \geq n/2+1\}}. \end{aligned} \quad (147)$$

By tail bounds for the Poisson distribution, we have $nP[N(n) < n/2 + 1] \rightarrow 0$ as $n \rightarrow \infty$, and hence, taking expectations in (147) and using (120), we obtain

$$E[|\mathcal{L}^\alpha(\mathcal{X}_n; C_n) - \mathcal{L}^\alpha(\mathcal{P}_n; C_n)|] = O(n^{\alpha(2\varepsilon+\sigma-1)+\varepsilon} \log n) + o(1),$$

which tends to zero since $\alpha \geq 1$, $1/2 < \sigma < 2/3$ and ε is small (see (129)). So we obtain the unrooted part of (146). The argument is the same in the rooted case. \square

Lemma 8.4 *Suppose \mathcal{X}_n and \mathcal{P}_n are coupled as described in Lemma 7.4, with $N(n) := \text{card}(\mathcal{P}_n)$. Let $\Delta(\infty)$ be given by Definition 4.1 with $H = \mathcal{L}^1$, and set $\alpha_1 := E[\Delta(\infty)]$. Then as $n \rightarrow \infty$ we have*

$$\mathcal{L}^1(\mathcal{P}_n; S_{0,n}) - \mathcal{L}^1(\mathcal{X}_n; S_{0,n}) - n^{-1/2} \alpha_1 (N(n) - n) \xrightarrow{L^2} 0; \quad (148)$$

$$\mathcal{L}^1(\mathcal{P}_n^0; S_{0,n}) - \mathcal{L}^1(\mathcal{X}_n^0; S_{0,n}) - n^{-1/2} \alpha_1 (N(n) - n) \xrightarrow{L^2} 0. \quad (149)$$

Proof. The proof of the first part (148) follows that of eqn (4.5) of [17], using our Lemma 4.5 and the fact that the functional \mathcal{L}^1 is homogeneous of order 1, is strongly stabilizing by Lemma 6.1, and satisfies the moments condition (62) by Lemma 6.3.

As shown in the proof of Corollary 6.1 (see in particular eqn (91)), we have that $\mathcal{L}^1(\mathcal{P}_n^0; S_{0,n}) - \mathcal{L}^1(\mathcal{P}_n; S_{0,n})$ converges to zero in L^2 and $\mathcal{L}^1(\mathcal{X}_n^0; S_{0,n}) - \mathcal{L}^1(\mathcal{X}_n; S_{0,n})$ converges to zero in L^2 . Therefore the second part (149) follows from (148). \square

We are now in a position to prove Theorem 2.2. We divide the proof into two cases: $\alpha \neq 1$ and $\alpha = 1$. In the latter case, to prove the result for the Poisson

process \mathcal{P}_n , we need to show that $\mathcal{L}^1(\mathcal{P}_n; B_n)$ and $\mathcal{L}^1(\mathcal{P}_n; S_{0,n})$ are asymptotically independent; likewise for \mathcal{P}_n^0 . We shall then obtain the results for the binomial process \mathcal{X}_n and for \mathcal{X}_n^0 from those for \mathcal{P}_n and \mathcal{P}_n^0 via the coupling described in Lemma 7.4.

Proof of Theorem 2.2 for $\alpha \neq 1$. First suppose $0 < \alpha < 1$. For the Poisson case, we have

$$\begin{aligned} n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n) &= n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; S_{0,n}) + n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n) \\ &\quad + n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; C_n). \end{aligned} \quad (150)$$

The first term in the right hand side of (150) converges in distribution to $\mathcal{N}(0, s_\alpha^2)$ by Theorem 6.1 (iv), and the other two terms converge in probability to 0 by eqns (105) and (135). Thus Slutsky's theorem yields the first (Poisson) part of (11). To obtain the second (binomial) part of (11), we use the coupling of Lemma 7.4. We write

$$\begin{aligned} n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n) &= n^{(\alpha-1)/2} \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; S_{0,n}) + n^{(\alpha-1)/2} (\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n \cup C_n)) \\ &\quad + n^{(\alpha-1)/2} (\tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; B_n \cup C_n) - \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n \cup C_n)). \end{aligned} \quad (151)$$

The first term in the right side of (151) is asymptotically $\mathcal{N}(0, t_\alpha^2)$ by Theorem 6.1 (ii). The second term tends to zero in probability by (105) and (135). The third term tends to zero in probability by (121). Thus we have the binomial case of (11).

The rooted case (8) is similar. Now, for the first (Poisson) part of (8), we use Corollary 6.1 (iv) with (106) and (136), and Slutsky's theorem. The second part of (8) follows from the analogous statement to (151) with the addition of the origin, using Corollary 6.1 (ii) with (106), (136), (122), and Slutsky's theorem again.

Next, suppose $\alpha > 1$. We have

$$\tilde{\mathcal{L}}^\alpha(\mathcal{P}_n) = \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; S_{0,n}) + \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; C_n) + \tilde{\mathcal{L}}^\alpha(\mathcal{P}_n; B_n). \quad (152)$$

The first term in the right hand side converges to 0 in probability, by Theorem 6.1 (iii). The second term also converges to 0 in probability, by the first part of (134). Then by (103) and Slutsky's theorem, we obtain the first (Poisson) part of (13). To obtain the rooted version, i.e. the first part of (10), we replace \mathcal{P}_n by \mathcal{P}_n^0 in (152), and combine (101) with Corollary 6.1 (iii) and the second part of (134), and apply Slutsky's theorem again.

To obtain the binomial versions of the results (10) and (13), we again make use of the coupling described in Lemma 7.4. We have

$$\tilde{\mathcal{L}}^\alpha(\mathcal{X}_n) = \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; S_{0,n}) + \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; C_n) + \tilde{\mathcal{L}}^\alpha(\mathcal{X}_n; B_n). \quad (153)$$

The first term in the right hand side converges in probability to zero by Theorem 6.1 (i). The second term converges in probability to zero by the first part of (134) and the first part of (146). The third part converges in distribution to $\tilde{F}_\alpha^{\{1\}} + \tilde{F}_\alpha^{\{2\}}$ by (104). Hence, Slutsky's theorem yields the binomial part of (13).

Similarly, by replacing \mathcal{P}_n by \mathcal{P}_n^0 and \mathcal{X}_n by \mathcal{X}_n^0 in (153), and using Corollary 6.1 (i), the second part of (134) and of (146), (102) and Slutsky's theorem, we obtain the binomial part of (10). This completes the proof for $\alpha \neq 1$.

Proof of Theorem 2.2 for $\alpha = 1$: the Poisson case. We now prove the first part of (9) and the first part of (12). Given n , set $q_n := 4\lfloor n^{\varepsilon+\sigma-1/2} \rfloor$. Split each cell Γ_i^x of C_n^x into $4q_n$ rectangular sub-cells, by splitting the horizontal edge into q_n segments and the vertical edge into 4 segments by a rectangular grid. Similarly, split each cell Γ_i^y by splitting the vertical edge into q_n segments and the horizontal edge into 4 segments. Finally, add a single square sub-cell in the top right-hand corner of C_n^0 , of side $(1/4)n^{\varepsilon-1/2}$, and denote this “the corner sub-cell”.

The total number of all such sub-cells is $1 + 8k_n q_n \sim 32n^{(1/2)-\varepsilon}$. Each of the sub-cells has width asymptotic to $(1/4)n^{\varepsilon-1/2}$ and height asymptotic to $(1/4)n^{\varepsilon-1/2}$, and so the area of each cell is asymptotic to $(1/16)n^{2\varepsilon-1}$. So for large n , for each of these sub-cells, the probability that it contains no point of \mathcal{P}_n is bounded by $\exp(-n^\varepsilon)$.

Let E_n be the event that each of the sub-cells described above contains at least one point of \mathcal{P}_n . Then

$$P[E_n^c] = O\left(n^{(1/2)-\varepsilon} \exp(-n^\varepsilon)\right) \rightarrow 0. \quad (154)$$

Suppose \mathbf{x} lies on the lower boundary of $S_{0,n}$. Consider the rectangular sub-cell of Γ_i^x lying just to the left of the sub-cell directly below \mathbf{x} (or the corner sub-cell if that lies just to the left of the sub-cell directly below \mathbf{x}). All points \mathbf{y} in this sub-cell satisfy $\mathbf{y} \preceq^* \mathbf{x}$, and for large n , satisfy $\|\mathbf{y} - \mathbf{x}\| < (3/4)n^{\varepsilon-1/2}$, whereas the nearest point to \mathbf{x} in B_n is at a distance at least $(3/4)n^{\varepsilon-1/2}$. Arguing similarly for \mathbf{x} on the left boundary of $S_{0,n}$, and using the triangle inequality, we see that if E_n occurs, no point in $S_{0,n}$ can be connected to any point in B_n , provided n is sufficiently large.

For simplicity of notation, set $X_n := \tilde{\mathcal{L}}^1(\mathcal{P}_n; B_n)$ and $Y_n := \tilde{\mathcal{L}}^1(\mathcal{P}_n; S_{0,n})$. Also, set $X := \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}}$ and $Y \sim \mathcal{N}(0, s_1^2)$, independent of X , with s_1 as given in Theorem 6.1. We know from Theorem 7.1 and Theorem 6.1 that $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{\mathcal{D}} Y$ as $n \rightarrow \infty$.

We need to show that $X_n + Y_n \xrightarrow{\mathcal{D}} X + Y$, where X and Y are independent random variables. We show this by convergence of the characteristic function,

$$E[\exp(it(X_n + Y_n))] \longrightarrow E[\exp(itX)]E[\exp(itY)]. \quad (155)$$

With ω denoting the configuration of points in C_n , we have

$$\begin{aligned} E[\exp(it(X_n + Y_n))] &= \int_{E_n} E[e^{itX_n} e^{itY_n} | \omega] dP(\omega) + E[e^{it(X_n + Y_n)} \mathbf{1}_{E_n^c}] \\ &= \int_{E_n} E[e^{itX_n}] E[e^{itY_n} | \omega] dP(\omega) + E[e^{it(X_n + Y_n)} \mathbf{1}_{E_n^c}], \end{aligned}$$

where we have used the fact that X_n and Y_n are conditionally independent, given $\omega \in E_n$, for n sufficiently large, and that X_n is independent of the configuration in

C_n . Then $E[e^{it(X_n+Y_n)}\mathbf{1}_{E_n^c}] \rightarrow 0$ as $n \rightarrow \infty$, since $P[E_n^c] \rightarrow 0$. So

$$E[\exp(it(X_n + Y_n))] - E[e^{itX_n}] E[e^{itY_n}\mathbf{1}_{E_n}] \rightarrow 0,$$

and we obtain (155) since $E[e^{itY_n}\mathbf{1}_{E_n}] = E[e^{itY_n}] - E[e^{itY_n}\mathbf{1}_{E_n^c}]$, $E[e^{itY_n}\mathbf{1}_{E_n^c}] \rightarrow 0$, $E[e^{itX_n}] \rightarrow E[e^{itX}]$, and $E[e^{itY_n}] \rightarrow E[e^{itY}]$ as $n \rightarrow \infty$.

We can now prove the first (Poisson) part of (12). We have the $\alpha = 1$ case of (152). The contribution from C_n converges in probability to 0 by the first part of (134). Slutsky's theorem and (155) then give the first (Poisson) part of (12). The rooted Poisson case (9) follows from the rooted version of (152), this time applying the argument for (155) taking $X_n := \tilde{\mathcal{L}}^1(\mathcal{P}_n^0; B_n)$, $Y_n := \tilde{\mathcal{L}}^1(\mathcal{P}_n^0; S_{0,n})$ and X, Y as before, and then using the second part of (134) and Slutsky's theorem again. Thus we obtain the first (Poisson) part of (9).

Proof of Theorem 2.2 for $\alpha = 1$: the binomial case. It remains for us to prove the second part of (9) and the second part of (12). To do this, we use the coupling of Lemma 7.4 once more. Considering first the unrooted case, we here set $X_n := \mathcal{L}^1(\mathcal{X}_n; B_n)$ and $Y_n := \mathcal{L}^1(\mathcal{X}_n; S_{0,n})$. Set $X'_n := \mathcal{L}^1(\mathcal{P}_n; B_n)$ and $Y'_n := \mathcal{L}^1(\mathcal{P}_n; S_{0,n})$ (note that all these random variables are uncentred).

Set $Y \sim \mathcal{N}(0, s_1^2)$ with s_1 as given in Theorem 6.1. Set $X := \tilde{D}_1^{\{1\}} + \tilde{D}_1^{\{2\}}$, independent of Y . Then by (155) we have (in our new notation)

$$X'_n - EX'_n + Y'_n - EY'_n \xrightarrow{\mathcal{D}} X + Y. \quad (156)$$

By (123), we have $X_n - X'_n \xrightarrow{P} 0$ and $EX_n - EX'_n \rightarrow 0$. Also, with α_1 as defined in Lemma 8.4, eqn (148) of that result gives us

$$Y'_n - Y_n - n^{-1/2}\alpha_1(N(n) - n) \xrightarrow{L^2} 0 \quad (157)$$

so that $E[Y'_n] - E[Y_n] \rightarrow 0$. Combining these observations with (156), and using Slutsky's theorem, we obtain

$$X_n - EX_n + Y_n - EY_n + n^{-1/2}\alpha_1(N(n) - n) \xrightarrow{\mathcal{D}} X + Y. \quad (158)$$

By Theorem 6.1 (iii) we have $\text{Var}(Y'_n) \rightarrow s_1^2$ as $n \rightarrow \infty$. By (157), and the independence of $N(n)$ and Y_n , we have

$$s_1^2 = \lim_{n \rightarrow \infty} \text{Var}[Y_n + n^{-1/2}\alpha_1(N(n) - n)] = \lim_{n \rightarrow \infty} (\text{Var}[Y_n] + \alpha_1^2) \quad (159)$$

so that $\alpha_1^2 \leq s_1^2$. Also, $n^{-1/2}\alpha_1(N(n) - n)$ is independent of $X_n + Y_n$, and asymptotically $\mathcal{N}(0, \alpha_1^2)$. Since the $\mathcal{N}(0, s^2)$ characteristic function is $\exp(-s^2 t^2/2)$, for all $t \in \mathbf{R}$ we obtain from (158) that

$$E[\exp(it(X_n - EX_n + Y_n - EY_n))] \rightarrow \exp(-(s_1^2 - \alpha_1^2)t^2/2)E[\exp(itX)]$$

so that

$$X_n - EX_n + Y_n - EY_n \xrightarrow{\mathcal{D}} X + W, \quad (160)$$

where $W \sim \mathcal{N}(0, s_1^2 - \alpha_1^2)$, and W is independent of X .

We have the $\alpha = 1$ case of (153). By the first part of (134) and the first part of (146), the contribution from C_n tends to zero in probability. Hence by (160) and Slutsky's theorem, we obtain the second (binomial) part of (12).

For the rooted case, we apply the argument for (160), now taking $X_n := \mathcal{L}^1(\mathcal{X}_n^0; B_n)$, $Y_n := \mathcal{L}^1(\mathcal{X}_n^0; S_{0,n})$, with X , Y and W as before. The rooted case of (156) follows from the rooted case of (155), and now we have $X_n - X'_n \xrightarrow{P} 0$ and $EX_n - EX'_n \rightarrow 0$ by (124). In the rooted case (157) still holds by (149), and then we obtain the rooted case of (160) as before.

To obtain the second (binomial) part of (9), we start with the rooted version of the $\alpha = 1$ case of (153). By the second part of (134) and of (146), the contribution from C_n tends to zero in probability. Hence by the rooted version of (160) and Slutsky's theorem, we obtain the second part of (9).

This completes the proof of the $\alpha = 1$ case, and hence the proof of Theorem 2.2 is complete. \square

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